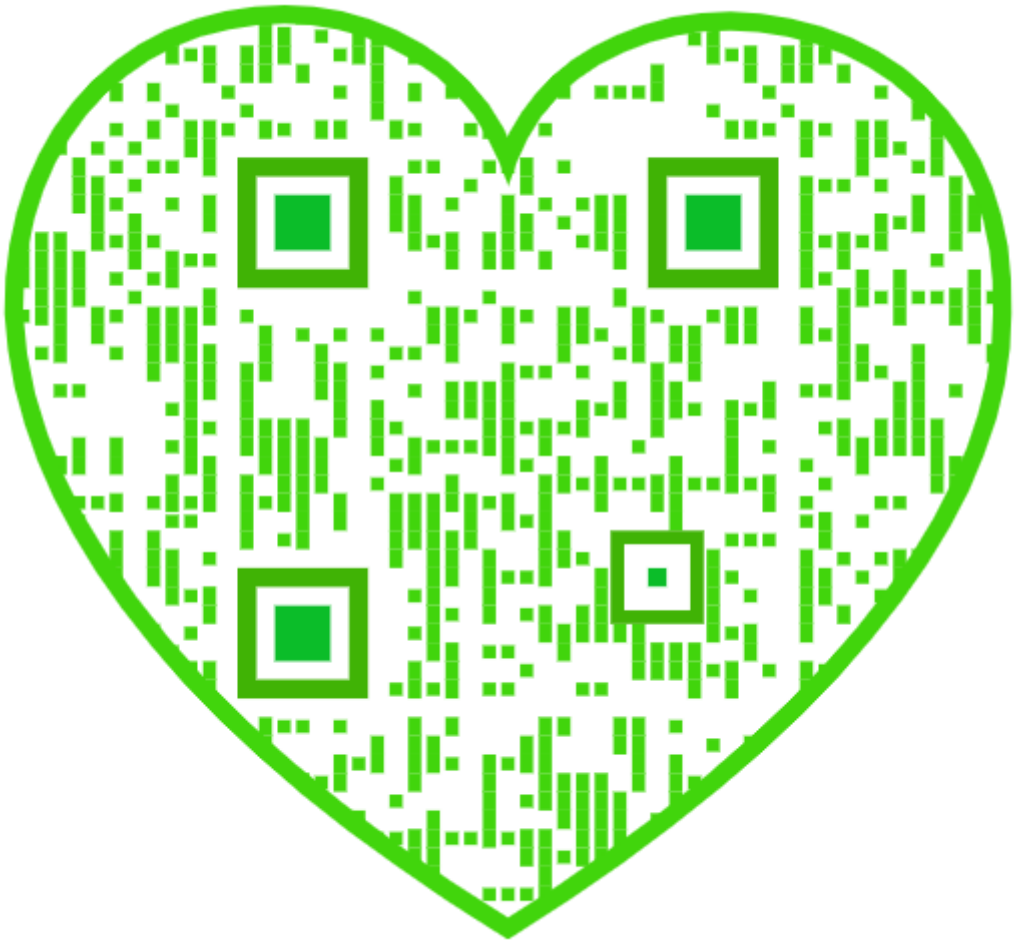

STEREOTOMY.

PROBLEMS

IN

STONE CUTTING.

IN FOUR CLASSES.

I. — PLANE-SIDED STRUCTURES.
II. — STRUCTURES CONTAINING DEVELOPABLE SURFACES.
III. — STRUCTURES CONTAINING WARPED SURFACES.
IV. — STRUCTURES CONTAINING DOUBLE-CURVED SURFACES.

FOR STUDENTS OF ENGINEERING AND ARCHITECTURE.

BY

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To all those Graduates of the

RENSSELAER POLYTECHNIC INSTITUTE,

in many successive years; who doubtless still retain a clear recollection, and, I hope, a pleasant remembrance of the "Oblique Arch," and the "Compound Wing-Wall,"

as drawn by them under my instruction;

This little Volume,

which contains improved illustrations of both, with many other problems, is most kindly and very respectfully dedicated by their former teacher and lasting friend

THE AUTHOR.

PREFACE.

THIS manual has been composed with the idea of representing, essentially, every class of structures, and every principal variety of surface, so as to make it most widely useful to the student in solving any other problems which he might meet.

The student needs, desires, and appreciates explicit detailed information in due abundance, not to *prevent* him from thinking for himself, but to *train* him to do so by examples fully explained. On this principle, and supported by the best authorities, I have discussed the few problems which could be admitted within the proposed limits, so thoroughly as to satisfy, I trust, all who enjoy the most — indeed, the only universally — available help, viz., a printed text.

I have attached scales and dimensions to the problems, which teachers and students may use or not, according as they prefer to work as if drawing actual structures for practical purposes, or to study the purely geometrical principles and operations involved. In either case, the figures should be made, generally, from two to three times as large as those of the plates in this volume, carefully following the text, and under frequent interrogation by the teacher, in doing so. I should add that this work presupposes a fair acquaintance with descriptive geometry, though many of its problems could be understood after the study of my "Elementary Projection Drawing." It is, however, complete in itself in regard to several collateral topics required for use in it, and not as conveniently found elsewhere.

I must here acknowledge my indebtedness to LEROY for suggestions of practical problems, and to ADHEMAR, in per-

fecting the treatment of the oblique arch. It seemed better to take examples essentially like actual structures, rather than imaginary ones, merely for the sake of greater apparent originality. But I have in every case made such changes, in various *details of design and treatment*, including, especially, the novel feature of numerous *examples for practice*, as to make my volume as much as possible a new contribution as well as a text-book. Moreover, problems VIII. and XV., and the systematic arrangement presented in the general table, will not be found elsewhere.

NEWTON, MASS., *May*, 1875.

NOTE TO TEACHERS.

Two complete successive courses, elementary and higher, can be made up from this work, as follows: —

Each problem to be drawn in Plan, Elevation, Section, Details, in isometric or oblique projection, and Developments.

I. Elementary Course. — *a. Plane-sided structures.* — 1. The buttressed walls (Prob. II.). 2. The plate band (Prob. III.). 3. A plane-sided wing-wall (Arts. 14–16).

b. Involving developable surfaces. — [See Arts. 17–28, and Probs. IV., V., and XVII. (the bracket), with their examples.] — 1. The segmental — 2. The full centred — 3. The sloping front — 4. The skew front — 5. The cylindrical-faced — 6. The rampant — 7. The conical recessed, arch. 8. The trumpet bracket.

c. Involving warped surfaces. — 1. Circular stairs around a central post. [See Arts. 119–125, and 126.] 2. The warped-faced wall (59).

d. Involving double-curved surfaces. — 1. The Niche (Prob. XVII.) 2. The dome only (Prob. XIX.).

II. Higher Course. — Any selection from the above, as introductory, for those who have not previously taken the foregoing *elementary course*, together with any of the other problems of each of the four *classes* noted in the table of contents.

GENERAL TABLE—PRACTICAL CLASSIFICATION.

STEREOTOMY. [STONE-CUTTING.]		Walls. Abutments. Piers. Buttresses.	Portals.	Arches.	Groins. Cloisters. Lunettes.	Wing-walls.	Trumpet Arches (Arched Corners)	Cupolas, Domes, and Niches.	Stairs.
GEOMETRICAL CLASSIFICATION. THE CHARACTERISTIC SURFACE.	**Plane.**	Wall and Buttress.*	Plate Band.*			Prismatic.* Pyramidal.*		Pyramidal Cupolas.	Straight Stairs.
	Developable.	Cylindrical and Conical Walls.	In a cylindrical,* or a sloping* wall; or in a *conical recess.*	With plane joints. { circular, elliptic, polycentral, etc.	Groined arches. { right. oblique. double. multiple. }* Cloisters.* Lunettes.*	Cylindrical and Conical (with plane tops) and { simple. compound.	Trumpets.*	Conical Cupolas.	
	Warped.	Piers with warped ice-breakers.	Warped gate recess.* (Marseilles gate.)	Oblique arches.* Horned Arch.* (Cor e de vache) Descending Ram- { direct pant { oblique.*		Hyperbolic paraboloidal. Cylindrical and Conical* (with warped tops). Conoidal.*		Hyperboloidal.	Circular.* Elliptic.* — St. Giles.
	Double Curved.		Hooded gate recess (St. Anthony's gate).*		Annular and* Radiant groin. — Fan vaults.			Niches. } Basins. }* Domes.* Pendentives.*	

* Illustrated in the text.

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STEREOTOMY.

STONE-CUTTING.

FIRST PRINCIPLES.

1. STEREOTOMY is that application of *Descriptive Geometry* which, comprehensively defined, treats of the cutting or shaping of forms, whether material or immaterial, so as to suit certain given conditions.

2. *Stereotomy, thus defined*, embraces, either by etymology, or established usage, the following subjects: —

1°. *Shades and Shadows*, or the *cutting of the volume of space* from which an opaque body excludes the light, by any *given surface*, on which the shadow of the body is thus said to fall.

2°. *Perspective*, or the *cutting of the cone*, of which the apparent limit of a given body is the base, and the eye the vertex, by any *given plane*, whose intersection with this cone is called the perspective of the given body.

3°. *Dialing*, or the cutting of metal plates so that their shadows upon a given surface shall mark the hours of the day.

4°. *Cinematics*, or the *shaping of mechanical forms*, so that by their mutual action they shall produce certain motions.

5°. *Structural articulations*, or the *shaping of the articulations* of wood and iron framings of every kind, with reference to convenience of construction and use.

6°. *Carpentry*, or the *cutting of wooden pieces*, so that when united they shall form a self-supporting whole.

7°. *Stone-cutting*, or the *cutting of stone pieces* of prescribed form, from the rough block, so that when combined in an assigned order, they shall form a *given or predetermined whole.*

Of these, the last two are the most obviously characteristic; that is, most clearly illustrative of the definition (1).

3. *Stone-cutting as a science* embraces three distinct parts: —

1°. The construction, on large scales in practice, of the *projections* of at least so much of a proposed structure as will permit —

2°. The derivation therefrom of the *directing instruments*, used by the workman as guides in cutting the rough block to its intended form by the chisel and mallet.

3°. The rules for the application of these directing instruments in the proper order and manner.

4. *The first two* of the three parts just mentioned consist of operations of *applied descriptive geometry*. The *number* of directing instruments, and the *mode* of their application, will depend considerably on the ingenuity of the designer.

5. *Practical stone-cutting*, or the actual formation of the finished stone, belongs to the student, only so far as it may, in the absence of models, serve him in gaining familiarity with those complex masonry forms which cannot be readily imagined from drawings alone.

In such modelling, the intended pieces would be wrought in plaster, by the aid of their wooden, or paper directors, derived from the drawings.

6. *Slopes are variously expressed.* 1°. In Pl. I., Fig. 1, the slope H″*k*, for example, may be expressed by the ratio, $\frac{\mathrm{T}k}{\mathrm{TH}''}$, = the tangent of the angle TH″*k*. Here $\frac{\mathrm{T}k}{\mathrm{TH}''} = \frac{5}{1}$; read, a slope of five to one.

2°. By degrees. Slopes of 30°, 40°, etc., make these angles with the horizontal plane.

3°. A batter of 1 inch to 1 foot, etc., means a horizontal departure of one inch from a vertical direction, for each foot of altitude.

4°. Nearly level slopes, as of railways, are described as a rise of 1 in 100, etc., 40 feet to the mile, etc.

5°. Once more; a slope of 45° being naturally described as that of 1 to 1, every other slope may be described by naming its *horizontal* component distance *first*, and by taking its *least* component as the *unit*. Thus a slope of 4 horizontal, to 1 vertical, may be described as a slope of 4 to 1. But one of 1 horizontal, to 7 vertical, for instance, as a slope of 1 to 7; or a batter of 1 in 7.

The nature of the case, or a reference to the figure, will show, in each problem, the meaning of any expression of slope that may be used.

Directing instruments.

7. The directing instruments (3) used in stone-cutting are of three kinds, *bevels*, *templets*, and *patterns*.

Bevels, as the common steel square, give the relative *positions* of required *lines*, or *surfaces* of a stone, by showing the angles between them. In the former case they are *plane* bevels; in the latter, *diedral* bevels.

Templets give the forms of required edges, or other distinguishing lines of a surface.

Patterns show the *forms* of *plane* or of *developable* surfaces. In the former case, they may be made of any stiff, thin material. In the latter, they must be flexible.

These instruments will be designated by numbers in the subsequent problems, and in every case, No. 1 will be a *straight-edge*, and No. 2 the *square*.

Notation.

8. For the sake of brevity, the horizontal and vertical planes of projection will, on account of the frequent reference which must be made to them, be denoted, respectively, as the planes **H** and **V**.

The usual rules for inking visible and invisible, given or auxiliary lines and planes (Des. Geom. 45), will be followed, unless in particular cases greater clearness may result from disregarding them.

When important lines are hidden by viewing a structure, as usual, vertically downwards from above it, its horizontal projection may be inked as if the object were seen by looking at it vertically upward from below it.

The greater complexity of some of the figures will make it convenient to adopt the rule of distinguishing invisible lines of the structure from the lines of construction, by dotting the former and marking the latter in short dashes; a distinction not shown, however, on the plates of this volume.

9. In order to secure a brief, yet comprehensive exhibition of the elements of stone-cutting, that is, one representing every important *class of structures*, and *form of surface*, they may be classified as in the General Table, the frontispiece.

CLASS I.

Plane-Sided Structures.

PROBLEM I.

To form plane surfaces of stone, making any given angle with each other.

This fundamental problem, being of constant occurrence, is here separately explained, in order to avoid repetition.

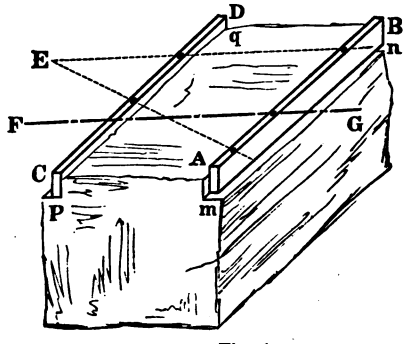


Fig. 1.

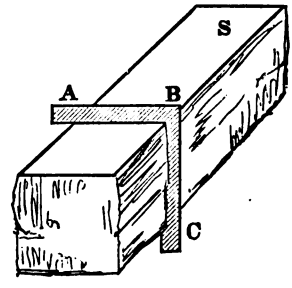


Fig. 2.

10. *First.* Fig. 1. represents the first steps in forming a plane upon a wholly unwrought block. Having two straight-edges, AB and CD, of equal width, ledges, as *m n* and *p q*, are cut on opposite edges of the stone, until the tops of the straight-edges placed on them, as shown, are found by sighting, as from E, to be in the same plane.

The portion of rough stone between *m n* and *p q* is then cut away until found, by frequent test with the straight-edge, applied transversely, as at FG, to be wrought down to the plane of *m n* and *p q*.

Second. Having prepared one plane surface, as shown in Fig. 2, a second plane face, perpendicular to the former, may be formed, as shown, by cutting two or more channels in any direction on the required face, until one arm, BC, of a *square* will fit any of them, while the other arm, AB, coincides with

the given face, and in a direction perpendicular to the common edge of the two surfaces. The intermediate rough stone is then cut down to the plane of these channels by applying the *straight-edge* transversely to them. For any other than a right angle, use a bevel giving such angle.

11. The *principle* of the *first* operation is, that if two lines, $m\ n$ and $p\ q$, are in the same plane, all lines, as FG, which intersect them, are in that plane also.

That of the *second* operation is, that if two planes are perpendicular to each other, any line, as AB, in one of them, and perpendicular to their intersection, will be perpendicular to *all* lines, BC, etc., drawn through its foot and in the other plane.

PROBLEM II.

A sloping wall and truncated pyramidal buttress.

I. *The Projections* (3). — These, Pl. I., Fig. 1, are made partly from given linear dimensions, and partly from given slopes of the inclined faces of the buttress and wall.

The plan and front elevation can be made wholly from data of the kinds just mentioned; but an end elevation is added, as a check upon errors, and as showing a different method of operation.

Let the wall be 8 ft. in height, and 3 ft. 6 in. thick at its base, and with a slope of 1 to 6 on its front. Let the vertical height of the buttress at c'' be 7 ft.; the slope of its front 1 to 5, that of its sides 1 to 4, and that of its top $4\frac{1}{2}$ to 1 (6). Then —

1°. *Construct the end elevation* by making $G''m = \frac{1}{6}$ of $G''E''$, and $A''C''$ parallel to $E''m$; also $E''N = 10$ ft., $ON = 2$ ft., and $H''k$ parallel to OE''; and $C''L = 4$ ft. 6 in., $LM = 1$ ft., and $a''c''$ parallel to MC'', having made $c''e$ 1 ft. below $C''E''$.

These operations will give the required slopes seen in the end view.

2°. *The construction of the plan and front elevation of the wall* is obvious on inspection, all the heights in elevation being determined by projecting across from the end view, and all the widths in plan, by transferring the horizontal distances, $E''C''$, etc., from the end view to any convenient line of reference, as e_1h_1, perpendicular to the ground line.

3°. *The projections of the buttress.* — The dimensions of its base are, JK = 6 ft.; HI = 4 ft.; and A_1U = 2 ft. 6 in. With these draw the base, whence, by means of the end view and the given slopes, all its lines can be found.

Thus, *ab* and *cd*, drawn indefinitely at first, are at distances, e_1b_1 and e_1d_1, from EF respectively equal to $a''f$ and $c''e$.

The slope of the sides being 1 to 4, draw *ij* parallel to HJ, and at a distance, *gi*, from it, equal to one fourth of G″E″. Then C*j* and *ij*, being in the plane of the top of the wall, J*j* is the intersection of the face of the wall with the left side of the buttress; and *c*, its intersection with d_1d, is a back upper corner of the buttress. Drawing the horizontal $a''f$, and making $e_1n_1 = fn''$, draw n_1n parallel to AB, and from *n*, draw *na* parallel to JH, and *na* will meet *ab* at *a*. Then draw *ac* and *a*H, which will complete the left side of the buttress. Its right side can be laid off from the axis of symmetry UV.

The vertical projection of the buttress is made by simply projecting the points of the plan up to the traces H′K′, $a'b'$, and $c'd'$ of the horizontal planes in which they lie, and then joining the points as shown.

In constructions like that of J*j*, the pairs of parallels, AJ and HJ; C*j* and *ij*, which determine the required line, should be as far apart as possible.

II. *The Directing Instruments.* — These are, besides Nos. 1 and 2 (7), No. 3, a pattern of the base of the upper stone of the buttress, and seen in its true size in plan: No. 4, a bevel containing the diedral angle, shown on the end elevation, between the base and the back of the buttress: No. 5, the like angle between the base and front of any buttress stone: No. 6, a bevel giving the angle between the base and either of the sides of the buttress.

No. 6 is found by revolving any line of greatest declivity, as R*h*, perpendicular to JH, until parallel to **V**, as at $Rh'' - R'h'''$; when it will show the true slope of the side of the buttress. No. 9 gives the angle between the top and back of the buttress.

If patterns are used as checks upon the operation of the bevels, they may be found as follows.

No. 7 is a pattern of the top of the buttress, and shown in its true form by revolution about $cd - c'd'$, till parallel to **V**. This is done by making $c'''V = a''c''$ and drawing $a'''b'''$ paral-

lel to $a'b'$ and limited by the vertical projections, $a'a'''$ and $b'b'''$, perpendicular to $c'd'$, of the arcs described by aa' and bb' in revolving about cd — $c'd'$.

No. 8 is a pattern of the right side of the upper abutment stone, and is shown in its true form by revolving it about PQ — P'Q' till horizontal. Then d'', the revolved position of dd', may be found by describing an arc with P as a centre and a radius equal to $d'P'''$, shown by the construction to be the true length of Pd — $P'd'$, and noting where it intersects dd'' perpendicular to the axis PQ. The point b'' being similarly found, $PQb''d''$ is the required pattern. These patterns would conveniently be open wooden frames.

III. *The Application.* — We will here illustrate this topic by the manner of working the *top stone of the buttress*. This, being the most irregular stone in the problem, the full explanation of the manner of forming it will enable the student to devise means for working any of the others.

Having selected a rough block in which the finished stone might be inscribed, first bring the intended base of the stone to a plane by Prob. I. Next scribe the edges of the base by pattern No. 3.

Having thus the bottom edges of all the lateral faces of the stone, these may be wrought from the base, and in their true relative position, by the bevels Nos. 4, 5, and 6. By marking the distances sc'' and ra'' on bevels Nos. 4 and 5 respectively, the edges ab — $a'b'$ and cd — $c'd'$ will be determined on the stone, which can then be finished by the use of No. 9 and the straight edge.

For increased accuracy, patterns Nos. 7 and 8, and others similarly found for the front and rear faces of the stone, can be used in determining the edges of the lateral faces more positively than by the natural intersection of these faces as found by the bevels and straight edge.

EXAMPLES. — 1°. Construct patterns 10 and 11, of the front and rear faces of the top buttress stone.

Ex. 2°. Construct a bevel, No. 12, giving the angle between the front, and an adjacent lateral face.

Ex. 3°. Show the construction and use of the guides necessary in working the stone below the top one of the buttress.

Ex. 4°. Do. for the top stone, supposing its top surfaces to be a pyramid having its vertex at vv'.

Ex. 5°. Construct, on a larger scale than in Fig. 5, a plan and front elevation

of the wall whose end elevation is there shown, with an isometrical figure of the stone NFEDC*q*. [The inclined surfaces, as *pq*, help to prevent sliding in the direction of the arrow, from the pressure of materials at the back, AN, of the wall.]

PROBLEM III.

The recessed flat arch, or plate-band.

12. An arch is an assemblage of blocks, mutually supporting, by means of radiating joints between them, and side supports to confine them laterally. When the arched surface, usually cylindrical, is plane, the structure is called a *Plate-band.*

I. *The Projections.* — Pl. I., Figs. 2, 3, and 4. The construction of these will, after a brief description, be sufficiently obvious on inspection of the figures, since, on all of them, the same letter indicates the same point.

T*t* J*j*—T′T″J′U′, Fig. 2, is the rectangular door-way, or opening through a wall. The door, not shown, folds against the vertical plane surface, U′V′J′Q′T″S″S′T′, which is in the plane *qs*. The recess which contains the door when shut, is bounded by the surfaces just pointed out, and by the three surfaces, Q*q* — V′Q′; Q*q*S*s* — Q′S″; and S*s* — S′S″, all of which are perpendicular to the plane V.

The surfaces PQ — V′W′P′Q′, PQRS — P′Q′R″S″, and RS — R′S′R″S″, collectively, form the convergent sides, or jambs of the door-way.

The joints, A′D′, etc., divide T″J′ into equal parts, and radiate from a point O, found by making OT″J′ an equilateral triangle. The lapping of the end stones, as Y, over the upper jamb stones, as at G′F′, is designed to give greater security.

Fig. 3 is an *oblique projection* of the stone A′D′E′F′, as seen in looking at it obliquely upward so as to see its front, right-hand, and under surfaces.* The identity of its several surfaces with the same ones as seen in Fig. 2, is shown by the lettering.

Fig. 4 is an *isometrical drawing* of the upper jamb-stone G′X′N′, showing its front, left-hand, and under surfaces.

II. *The Directing Instruments.* — Of these the only ones requiring any further graphical operations for their construction

* See my ELEMENTARY PROJECTION DRAWING: Div. IV. Chap. V., for an explanation of this species of projection.

are the patterns of the radiating joints. One of these is shown at $l'p'p''f''f'''k'''k''$, by revolving the joint, vertically projected in $f'p'$, about that line, considered as in the plane RX, as an axis; whence $p'p''$=Rr; $k'k''$=cu; $k'k'''$=cA, etc.

The true form of the joint on J′G′ may be similarly found.

III. *The Application.* — Let it be required to work the stone A′D′F′, Figs. 2 and 3. There is no one precise order of operations which is indispensable; but we should naturally work first, either the top surface, E*e*D*d*, as the *simplest*, or the back *deaj*, or joint on A′D′, as the *largest and most important.*

Supposing then that we have a block in which the stone can be inscribed, as shown at *f'hmo;* bring the intended joint surface on A′D′ to an indefinite plane by Prob. I., and by the pattern, No. 3, mark its edges. Next, the back and front can both be wrought square with this surface by No. 2, and their edges marked by their patterns A′D′E′F′G′J′ (No. 6), and C′D′E′F′G′H′ (No. 7). The top and bottom surfaces can then be wrought, like all the lateral surfaces, either square with the back, by No. 2; or, from the joint surface A′D′ by the bevels Nos. 4 and 5.

The surfaces on E′F′ and G′F′ are square with each other and with the front and back. A pattern, No. 8 (not shown), of the joint on J′G′, and a bevel, No. 9, set to the angle QPX, between the vertical and jamb surfaces, will suffice for the accurate completion of the stone.

13. The stones A′D′F′ and G′F′N′ cannot be more clearly shown than in Figs. 3 and 4, in which all their most irregular portions are made visible. But the student cannot here become too familiar with the methods of representation there given, on account of their special usefulness in the drawing of complicated masonry. Hence the nature of a part of the following examples.

Ex. 1°. Make an isometric drawing of the stone Y, as inscribed in a prism containing its top and bottom faces.

Ex. 2°. Make the same drawing of Y as inscribed in the prism *f'hmo.*

Ex. 3°. Make an oblique projection of the upper *left-hand* jamb-stone.

Ex. 4°. Draw the instruments, and describe their use, for working the upper jamb stones as G′F′N′.

Ex. 5°. Devise and describe any other convenient manner of working the stone Y, than that given in the text.

Ex. 6°. Make the top of the door-way semi-octagonal, by sloping the under sides of Y and A′D′G′ at 45° to the vertical sides, on T*t* and J*j*, of the opening.

Plane Sided Wing-Walls.

14. *Wing-walls*, are those which flank the approach to any passage-way, or area, so as to prevent obstructing it.

15. Pl. I., Figs. 6–9, illustrate in outline the leading forms of plane sided wing-walls; *rectilinear*, *prismatic*, and *pyramidal.*

Fig. 6, a rectilinear wing-wall, AB — A′B′, is such as might be used to retain a roadway ascending behind it, and bending, as shown by the arrows, to cross a bridge, one of whose abutments is CC′. Such a construction is often seen where a railway or canal is crossed at right angles by a road which, on one or both sides of the crossing, is parallel to the railway or canal.

Fig. 7 represents wing-walls, as *abc* — *b′a′c′*, which, with the arch-wall, *cd* — *c′d′*, form three sides of a vertical hollow quadrangular *prism*, in which the tops of the wing-walls are truncated by an oblique plane, *a′c′d′g′*. Had there been a slope or batter to the arch and wing-walls, the inner surfaces, collectively, would have formed part of a vertical quadrangular *pyramid.* Either construction would serve in case of a railway tunnel occupying the line of a city street.

Fig. 8 represents wing-walls, which, with the connecting arch-wall, DF, form half of a vertical hollow hexagonal prism, wholly truncated by an oblique plane, AEG — A′E′G′; and flanking the inlet to a pipe culvert, *mn* — *m′n′*, by which the water of some gulley or small ravine may be led through an embankment.

Fig. 9 differs from Fig. 8, in that the top of the arch-wall, *dstg* — *d′g′*, is horizontal, and the inner faces of the walls form a part of a vertical inverted *pyramid.*

16. *A coping* is a layer of thin stones, laid on the top of a wall, and made broader than the thickness of the wall, which they cover, for the purpose of sheltering the joints from the weather, while their adaptation to this end and the effect of their shadows, add to the beauty of the whole.

Had the arch-wall, Fig. 9, been truncated by the horizontal plane P′Q′, it might have received a coping, whose front edge would have been, in plan, at *pq*, formed by projecting *r′* at *r*. Lines between, and parallel to *be* and *fk*, would represent the front of the copings of the wing-walls.

Problems I. — III. will enable the student to work the following —

EXAMPLES. — 1°. Draw Pl. I., Fig. 7, to scale from measurements, with a section on the plane nNn', and describe the cutting of a stone at the corner dd', which, to break joints with courses below, shall extend from kf in the arch-wall, cd, to p_1 in the wing-wall, ee'.

Ex. 2°. Draw Pl. I., Fig. 8, in like manner, making the vertical joint surfaces, perpendicular to FD and FG, and describe, with an isometrical or oblique projection, the working of the top stones at one of the angles D or F.

Ex. 3°. In Ex. 2°, let there be a coping, and let the top be truncated as at $P'Q'$, Fig. 9.

Ex. 4°. Treat Pl. I., Fig. 9, as described in Ex. 2°.

Ex. 5°. In Ex. 4, let there be a coping. Also add square piers at the foot of the wing-walls, each side of which in plan shall be equal to ac, and whose height shall be $a'a''$.

CLASS II.

Structures containing Developable Surfaces.

17. *Walls* having either *cylindrical* or *conical* faces, will be treated incidentally, in connection with other constructions of which they form a part.

Arches (12) include as particular cases, *portals* or arched openings through walls; and combined, or *groined*, and *cloistered arches*. *Vaults* include, more comprehensively, domes, and other over-arched areas.

Only cylindrical arches will now be considered, and under the heads of, —

I°. Definitions. III°. Preliminary Constructions.
II°. Classification. IV°. Practical Illustrations.

I°. — *Arches* — *Definitions.*

18. *Arch Masses.* — The supporting walls of an arch are called its *abutments* when backed by earth, etc., and *piers*, when exposed on all sides. The superincumbent masonry, supported by the arch, is called the *spandril*, or *backing*.

Each row of blocks extending through the length of the arch is a *course*. Each block in the course is an *arch-stone*, or *voussoir*. The extreme *voussoirs* of each course are the *quoins*, or *ring-stones*. The middle ring-stone is the *key-stone*.

19. *Arch Surfaces.* — The end surfaces of an arch are its *faces*, $a'd'$ — $aQDd$, Pl. III., Fig. 27. (In certain cases, where the elevation is more important, or naturally made first, the accented letters are attached to the plan (8)). The inner cylindrical surface, ABD, is the *intrados*. The outer one, whether cylindrical, Fig. 29, or plane-sided, Fig. 27, is the *extrados*. The top of the abutment, if level, is called the *springing plane;* if inclined radially, the *skew-back*.

The longer sides, $mm'rr'$, Fig. 26, of a voussoir, are its *beds*, and are continuous from stone to stone. Its ends, are its *heads*, and these terminate, or break, in the coursing surfaces. Its inner side is its *soffit;* the outer, its *back*.

20. *Arch Lines.* — The intersections of the faces with the intrados and extrados are the *face lines* of the arch, and of those surfaces. The intersections of the springing plane, or the skew-back, with the same surfaces, are the *springing lines.* The *axis* joins the centres of the face lines, and is parallel to the springing lines in an arch of uniform cross section, and is also straight in a cylindrical, and a conical arch. The edges, parallel to the axis, of the courses are the *coursing joints;* the transverse edges of the voussoirs are the *heading joints;* the edges lying in the thickness of the arch are the *radial joints,* and the joints in the face are the *face joints.* The *span* is the perpendicular distance between the springing lines. The *rise* is the greatest height of the intrados above the span.

21. *Arch Points.* — The highest point of any right section of an arch is its *crown.* The intersection of the axis with the face is the *face centre.* When the face joints radiate from any other point than the face centre, such point is called a *focus.*

II°. — *Arches — Classification.*

22. *Arches may be classified* according to either, 1°, the *form of the intrados;* 2°, the *relation of the axis to other parts;* or 3°, the *forms of the face lines.*

The two former are the most important grounds of division, since they give rise to more radical differences of design. The latter occasions only the many minor modifications of form.

23. ARCHES, divided according to the *forms of their intrados,* are *plane, developable, warped,* or *double-curved.*

Plane arches have been illustrated in Prob. III.

Developable arches are those in which the intrados is nearly always cylindrical.

Warped and double-curved arches will be treated in connection with other problems involving like surfaces.

24. ARCHES, divided according to the *direction of the axis,* relative to other parts, are *right, oblique, descending,* and *rampant.*

A right arch is one in which the axis is *horizontal,* and the planes of the faces at least so nearly perpendicular to it that the *coursing joints can all be parallel to the axis.* The face may be oblique to **H**, when it has a *batter;* or oblique to a vertical plane through the axis, when it is a *skew*-face. It may be oblique to both of these planes at once.

25. The *oblique arch*, properly so called, is one in which the coursing joints, in order to be nearly or quite perpendicular to the face, are not parallel to the axis, the latter being horizontal, and oblique to the horizontal lines of the face.

26. *A descending arch* is one whose axis is inclined to a horizontal plane.

A rampant arch is one in which one springing line is higher than the other. The descending arch is also sometimes called rampant. The two cases may then be distinguished as *longitudinally rampant*, when *one end of the axis* is higher than the other, and *transversely rampant*, when one *springing line* is higher than the other.

27. ARCHES, divided according to the *form of the face line*, may be *circular*, *elliptic*, *parabolic*, etc., according to the form of the right section of the intrados. They are also *depressed*, Pl. II., Figs. 13, 15, or *super-elevated*, Fig. 14, according as the rise (20) is less or more than half of the span. They are also *pointed*, if, as in Fig. 14, the right section is composed of intersecting arcs.

When the right section of an arch is a semicircle, semi-ellipse, etc., the arch is said to be *full-centred*. When this curve is less than a half one, the arch is called *segmental*.

28. When the right section is compound, having three, five, or many centres, the arch is said to be *three-centred*, etc., or *polycentral*, Pl. III., Figs. 11, 21.

III°. — *Arches — Preliminary Constructions.*

§ *Conic Sections.*

1°. — To construct a circle by points, having given its radius.

29. Let O*e*, Pl. II., Fig. 10, be the radius of the required circle, and $cd = 2Oe$ the diameter perpendicular to O*e*. Draw *ep* and *cp*, which, with O*c* and O*e*, will form a square, and hence be tangent at *c* and *e*. Divide O*e* and *ep* into the same number of equal parts, numbered as in the figure. Then lines radiating from *c* and *d*, through the like points on *ep* and O*e*, respectively, will meet at points *v*, *r*, etc., of the circle whose radius is O*c*. For O*fd* and *cgp* are equal right triangles, and the angle O*df* is moreover common to the triangles O*fd* and

dnc.

~~Onc.~~ Then by subtraction, the angles *ncd* and O*fd* are equal, which makes the triangles O*fd* and *dcn*, similar, and hence *dcn* right angled at *n*. Hence *n* is a point of the semicircle whose diameter is *cd*.

2°.—To construct an arc of a circle by points, knowing its chord and versed-sine or rise.

30. Let 2*cb*, Pl. II., Fig. 15, be the given chord, and *ab* the given rise of the required arc. Draw *ac*, and *ce* perpendicular to it, and limited by the tangent *ae* at *a*; also *cn* parallel to *ab*. Then divide *cb*, *ae*, and *cn*, each into the same number of equal parts, and number them as in the figure. Then lines joining like points on *cb* and *ae*, will meet those radiating from *a* to the points on *cn* in points of the required arc.

Parallels to O*c* and O*e*, Fig. 10, from any point of the arc *ce*, will show the correctness of this construction, and of points as *k* on the arc *ac*, Fig. 15, produced.

Making *cc'* radial and equal to *aa'* we may likewise, as seen in the figure, construct the concentric arc *a'c'* of the extrados of a segmental arch (27).

31. *The radius of the segmental arch*, given by its span and rise, may be desired, and may be found thus. Pl. II., Fig. 15.

Let C, intersection of *ab*C and *r*C, the bisecting perpendicular to *ac*, be the centre of the arc *ac*. Then let R = C*a*; $s = bc$ and $v = ab$. Now $ab : ac :: ar : a\mathrm{C}$.

$$\text{Whence } a\mathrm{C} = \frac{ac \times ar}{ab}.$$

$$\text{But } ar = \tfrac{1}{2}\, ac; \text{ hence}$$

$$a\mathrm{C} = \frac{ac^2}{2ab} = \frac{s^2 + v^2}{2v}.$$

3°.—To construct an ellipse by points on given axes. Also, normals to it.

32. Let AB and CD be the given axes, Pl. II., Fig. 16. Then as the projection of a circle seen obliquely is an ellipse (Des. Geom., Theor. VI.), the curve may be found as at *a*, *b*, *c*, as in Fig. 10, as is evident by comparing Figs. 10 and 16.

In an elliptic arch the face joints should be normal to the elliptic face line of the intrados.

33. *Construction of normal Face Joints. First Method.*—

Let N be a point at which such a joint is to be drawn. With C or D as a centre, and a radius equal to A*h*, describe arcs intersecting AB at F and F′, which are the foci of the ellipse. Then the normal N*m*, at N, bisects the angle FNF′.

34. *Second Method.* — The normal at any point being perpendicular to the tangent at the same point, to draw the normal at T, for example; describe an arc as A*t*, with centre *h*, and radius A*h*, and produce *d*T, perpendicular to AB, to meet this arc at *t*. This arc may represent the circle, which, when revolved about AB till oblique to the paper, is projected in the given ellipse. Then draw the tangent *t*K to the arc, and as K, being in the axis AB, remains fixed, KT is the tangent to the ellipse at T. Then T*k*, perpendicular to KT, at T, is the required normal at T.

4°. — To construct the arc of a parabola; on a given segment of the axis, and the chord which is perpendicular to the axis. Also, normals to it.

35. On comparing Figs. 10, 16, and 20, Pl. II., and knowing from geometry (Des. Geom., Arts. 206–209), that the several conic sections have certain general properties in common, their construction may be put in general terms applicable to all cases, thus. Points of any conic section are found at the intersection of sets of lines which radiate from the two vertices of an axis of the curve; those from one vertex radiating to points of equal division on the common perpendicular to the chord and to the tangent at that vertex; and those from the other vertex, to like points on the semi-chord, in the manner shown in the figures. Thus, Pl. II., Fig. 20, let AU be a semi-chord, and BU a segment of the axis of a required parabola. Draw the tangent BV, limited by AV, parallel to BU. Divide AV and AU into the same number of equal parts, and number like points, reckoned from A, with like numbers. Then radials from B to the points on AV, will meet radials from the opposite vertex (B′) of the axis BU to like points on AU, in points, *k*, *c*, N, etc., of the parabola. But note that as the axis is of infinite length, the radials from its infinitely distant extremity, B′, will be parallel to BU as shown at N*m*, etc.

The figure further shows a depressed gothic arch, each half composed of parabolic arcs, as AB and DE.

36. *Construction of the Normal Joints.* — Let such a joint

be constructed at N. Draw NN′ perpendicular to the axis BU, make BT = BN′, and TN will be the tangent at N by the property that the subtangent, N′T, is bisected by the vertex, B, of the curve. Then N*n*, the normal, is perpendicular to the tangent NT.

In the parabola, the sub-normal N′*n* is constant; hence, to construct any other normal, as at *e*, we have only to draw *ee′* perpendicular to BU, make *e′h′* = N′*n*, and *h′e* will be the normal at *e*.

37. Pl. II., Fig. 13, represents a depressed gothic arch formed of circular arcs, in imitation of Fig. 20. The construction (Draft. Insts. & Oper. p. 91) may be obvious on inspection. The joints in each arc radiate to the centre of that arc. Fig. 14 represents a pointed or lancet gothic arch, where the radii, as *ca* and *cb*, are greater than the span, *de*.

5.—To construct an arc of a hyperbola, on a given chord, and segment of the axis perpendicular to the chord.

38. This form of the conic section is seldom needed by the mason. Yet to complete the series of similar constructions it is here given. Let *aa′*, Pl. II., Fig. 18, be the chord, and V*e* the segment of the axis. Make *a′t′* and *at* equal and parallel to V*e*, and divide *at* and *ae* similarly, as in the previous examples. Then, WV being that axis of the hyperbola which is perpendicular to the given chord *aa′*, radials from V to the points on *at* will meet those from W, to the like numbered points on *ae*, in points as *b*, *c*, *d*, of the required hyperbola.

39. This construction is essentially the *perspective* of that of Fig. 10 on a plane so placed as to cut the visual cone whose base is the circle, in a hyperbola; while that of Fig. 20 is similarly the perspective of Fig. 10 upon a plane cutting the visual cone whose base is the circle, in a parabola.

40. In Figs. 16, 18, and 20, we may substitute *any* chord and its conjugate diameter (that is, the one which bisects the given chord and its parallel tangent) for the axis and a chord perpendicular to it; since such constructions would merely be the projections, or perspectives of Fig. 10 on planes not parallel to either of the given diameters, O*c* and O*e*.

§§ *Polycentral Arch-Curves.*

41. The practical convenience of circular, above that of other curved work, with other reasons which will appear, has led to the frequent adoption of artificial, or compound curves, composed of circular arcs of different radii, in place of the true conic sections; especially in case of separate elliptic arches, *i. e.* when *not* combined as in groined arches. And as there will be found to be a choice in the relative lengths of these radii, they should not be chosen arbitrarily, but so as to give the best proportions to the intended oval, as will next be shown.

Three-Centred Ovals.

1°. To construct the general case of the semi-oval of three centres.

42. In Pl. II., Fig. 17, let ab and bg be the given semi-axes of the proposed curve. Assume ac at pleasure, but less than bg; make $ge = ac$, and draw ce; bisect ce by the perpendicular to it, fd, meeting gb produced at d. Then will c, d, and a point c_1, taken on ab produced so that $bc_1 = bc$, be the three centres of the required oval.

43. From this beginning, the following investigation leads to useful special cases. Let

$$ab = a\,;\ bg = b\,;\ ac = r\,;\ dg = \mathrm{R}.$$

Then from the triangle bcd we have,

$$(cd)^2 = (bd)^2 + (bc)^2.$$

That is $$(\mathrm{R} - r)^2 = (\mathrm{R} - b)^2 + (a - r)^2 \qquad (1)$$

whence $$\mathrm{R} = \frac{a^2 + b^2 - 2ar}{2\,(b - r)} \qquad (2)$$

and $$r = \frac{a^2 + b^2 - 2\,bR}{2\,(a - R)} \qquad (3)$$

equations which obviously allow an infinite number of solutions for the same given axes; as the construction shows, where $ac, = r$, was assumed at pleasure, only less than bg; and thence R was found.

2°. First special case. The semi-oval of three centres, when the lesser arc is 60°.

44. Let the arc am, Pl. II., Fig. 11, $= 60°$; whence $bcd = cdc' = 60°$, and the arc $mn = 60°$. Now put $r = a - x$, where $bc = x$, then $\mathrm{R} = md = ac' = a + x$.

Substituting these values of R and r in (1) Art. 43, we find

$$x^2 - (a-b)x = \frac{(a-b)^2}{2}$$

whence, neglecting the negative value of x,

$$x = \frac{a-b}{2} + \frac{a-b}{2}\sqrt{3}, \qquad (4)$$

which is *constructed as follows:*

Take $bf = a - b$, $bh = \frac{a-b}{2}$, and on fh describe a semi-circle; in which $(bk)^2 = bh \times bf$. Then describe the arc kc from h as a centre, which will give $bc = x$. Then make $bc' = bc$ and cdc' an equilateral triangle on cc', which will give the three centres required.

For, as $bf = 2\,bh$; $(bk)^2 = bh \times 2bh = 2(bh)^2$

and $(hk)^2 = (bh)^2 + (bk)^2 = 3(bh)^2$

or, $(hk) = (bh)\sqrt{3} = \frac{(a-b)}{2}\sqrt{3}$;

then $bc = bh + hk = \frac{a-b}{2} + \frac{a-b}{2}\sqrt{3}, = x$; and

$r = a - x$, and $R = a + x$ as required.

45. The curve in Fig. 11 has the advantage, in application to bridge arches, that when b is $> \frac{2}{3}\,a$, it affords a greater interior capacity for the flow of water than does an ellipse on the same axes. For the radius of curvature at a, Fig. 17, of the ellipse on ab and bg as semi-axes, is $r_1 = \frac{b^2}{a}$, but

$$r = a - x = a - \left(\frac{a-b}{2} + \frac{a-b}{2}\sqrt{3}\right)$$

Now when $b = \frac{2}{3}\,a$,

$$r_1 = \frac{\frac{4}{9}a^2}{a} = \frac{4}{9}a = .444\,a.$$

$$\text{but } r = \left(1 - \frac{1+\sqrt{3}}{6}\right)a = .545\,a.$$

3°. Second special case. The ratio $\frac{R}{r}$ to be a minimum.

46. From Eq. (2) Art. 43, $\frac{R}{r} = \frac{a^2 + b^2 - 2ar}{2r\,(b-r)}$

Differentiating by the rule for fractions (a and b, constants) dividing by dr, and placing the result $= 0$, we have

$$\frac{du}{dr} = \frac{d\,\frac{R}{r}}{dr} = ar^2 - (a^2 + b^2)\,r + \tfrac{1}{2}b\,(a^2 + b^2) = 0;$$

which solved with respect to r gives, after reducing, and neglecting that value of r which makes $r > b$,

$$r = \frac{\sqrt{a^2+b^2}}{a}\left(\frac{\sqrt{a^2+b^2}-(a-b)}{2}\right) \qquad (5)$$

Equating this with (3) and reducing,

$$\mathrm{R} = \frac{\sqrt{a^2+b^2}}{b}\left(\frac{\sqrt{a^2+b^2}+(a-b)}{2}\right) \qquad (6)$$

The direct reduction being somewhat tedious, note the symmetry of Eqs. (2) and (3), where (3) is obtained by substituting a for b and R for r in (2); and it will be obvious that (6) is obtained from (5) by a like substitution.

47. *The construction*, which is very simple, is shown in Pl. II., Fig. 19. Draw the chord ae, make $ef = a - b$, bisect af by the perpendicular gd, meeting eb produced, and ab, at d and c, the required centres for the half curve aoe. For, the similar triangles agc, aeb, and gde give

$$r = ac = \frac{ae}{ab} \times ag = \frac{\sqrt{a^2+b^2}}{a}\left(\frac{\sqrt{a^2+b^2}-(a-b)}{2}\right)$$

and $$\mathrm{R} = de = \frac{ae}{be} \times eg = \frac{\sqrt{a^2+b^2}}{b}\left(\frac{\sqrt{a^2+b^2}+(a-b)}{2}\right)$$

Five-Centred Ovals.

4°. To construct a five-centred semi-oval, which shall conform as nearly as possible to a semi-ellipse on the same axes.

48. Five-centred ovals are preferable to three centred ones, when $\frac{b}{2a} < \frac{1}{3}$; and are generally most pleasing when, as here required, they most nearly resemble an ellipse, described on the same axes. In Pl. II., Fig. 21, let $fg = \frac{1}{2}\,ag$. It is a property of the ellipse that its radius of curvature, at the extremity of the minor axis, is a third proportional to the semi-minor and semi-major axes. Hence make $fe = 2ga$ and e is one of the five centres. Again, the radius of curvature at the extremity of the major axis, is a third proportional to the semi-major and semi-minor axes, hence, make $ca = c'b = \frac{1}{2}fg$, and c and c' will be two other centres. Now, since the radius of curvature of an ellipse is changing continually, a radius may be found which shall be a mean proportional between the radii already found, and such a radius is also a mean proportional between the semi-axes, for $fe : ga :: gf : ac$; $\therefore fe \times ac = ga \times gf$

that is, $R \times r = a \times b$, and calling the intermediate radius r_1, we have by hypothesis,

$$r_1^2 = R \,.\, r. \quad \text{Hence } r_1^2 = a \,.\, b.$$

that is, $a : r_1 :: r_1 : b$.

Hence make $gj = gf$, describe a semicircle on aj; draw gk, and we have gk, a mean proportional between $ag, = a$, and $gj, = b$, and hence equal to the radius r_1. Now make $if = gk$; draw the arc did'; make $ah = gk$; draw the arc hd, and similarly draw $h'd'$; then d and d' are the remaining centres; for fe may be regarded as containing all the lesser radii found in moving from f to a about e as a centre; *i. e.*, at some point of the motion of ef about e, it becomes $= d\ m$. Likewise ab may be regarded as containing all the radii greater than ac, found in moving ab about c as centre, *i. e.*, at some point of the motion of ab about c, ac becomes $= d\ m$, hence d must be at the intersection of arcs di and dh, drawn with e and c respectively as centres, ah and fi each being equal to r_1. A general construction of the five-centred oval is given in my "Drafting Instruments and Operations," p. 92.

5°. To construct the five-centred oval, by a method applicable to an oval having any number of centres.

49. See Pl. II., Fig. 12, where, as before, put $ao = a$, and $bo = b$. The problem is indeterminate when the extreme radii R and r are both chosen arbitrarily, for calling $r_1 =$ the intermediate radius, we have $cc' = r_1 - r$, and $c''c' = R - r_1$; making but two equations for the three unknown quantities cc', $c''c'$, and r_1. Note that as $cc' + c'c'' = R - r$, the centre c' will be on an ellipse whose foci are c and c'', and whose major axis is $R - r$.

In order to render the problem determinate, put

$$R - b = 3\,(a - r), \text{ then, } \begin{cases} 1°, & oc'' = 3\ oc, \textit{ i. e.} \\ 2°, & ec'' = \tfrac{1}{2}\ oc'', \text{ and} \\ 3°, & cd = \tfrac{1}{3}\ co. \end{cases}$$

The problem can now be solved geometrically, and without assuming any of the required radii, thus. Assume ag; make $og'' = 3og$; bisect og'' at e'; join g and e', and with g as a centre, and radius ag, describe the arc aG. Then take $dg = \frac{1}{3}\ og$; draw dg'', which will cut ge' in a point g', which is the centre of the arc GV; and g'' is the centre of the arc VC. In general, however, it is useless to draw the arcs as these will not be, except by accident, the ones required.

But the polygons, $ogg'g''$, and $occ'c''$, having their sides made proportional, will always be similar, hence if we put,

$$oc = x\,;\ og = p\,;\ cc'+c'c'' = z\,;$$
$$oc'' = y\,;\ og'' = q\,;\ gg'+g'g'' = s,$$

we shall have

$$\frac{x}{p}=\frac{y}{q}\,;\ \frac{x}{p}=\frac{z}{s}\,;\ \text{and } z+a-x=y+b,$$

from which after eliminating z,

$$x=\frac{(a-b)\,p}{p+q-s} \text{ and } y=\frac{(a-b)\,q}{p+q-s}$$

or, when as was assumed, $q=3\,p$, then $y=3x$, and

$$x=\frac{(a-b)p}{4p-s} \qquad (1) \qquad \text{and } y=\frac{(a-b)\,3p}{4p-s} \qquad (2)$$

IV°.— *Arches — Illustrations.*

Problem IV.

A three-centred arch in a circular wall.

I. *The Projections.* In Pl. III., Fig. 22, $a'a''d'd''$ is the plan of a segment of the wall, 4 feet 7 inches thick, of a circular room, which has a radius, $C'b$ $(=bb'')$ of 25 feet 8 inches. This wall is pierced by a horizontal cylinder, whose axis $C''C'-C$, intersects the vertical line, at b, that of the room, at b, C, and whose right section is ABD; forming an arch, whose span, AD, is 17 feet 4 inches; rise, CB, is 6 feet 2 inches; interior height, GB, is 17 feet 2 inches; and height to top of keystone is 20 feet 2 inches.

The face line, ABD, is found by (47) the joints radiate to the centres O, c' and c'', and are adjusted to the horizontal joints of *eleven* equal courses of stone, each 22 inches thick.

From these numerical data, and general descriptions, the drawings can be made (better on at least a scale of $\frac{1}{10}$) and, in the plan, should show *all* the coursing joints (20) of one half of the intrados, as e''' is shown at e^4e^5.

II. *The Directing Instruments.* In illustration of the derivation of these from the projections of the arch, we shall consider some one stone. Let it be *mnpqr*. As usual, the instruments required will be of two kinds: *patterns* to deter-

mine the *forms* of the faces of the stone; and *bevels* to determine their relative *positions* (7).

1°. The pattern, No. 3, of the top, pq, will be the figure $P'P''Q'Q''$, seen in its real size in the plan.

2°. That of the side on qr, No. 4, will be simply a rectangle, of width qr, and length $Q\,Q''$.

3°. That of the radial bed on mr, No. 7, requires the *construction of the true form of the elliptical face-joint*, mr — $M'Q'$. As this construction is the same for all such joints, it is here given for them all. (See Des. Geometry, Part I, Problem LXXXIX., 2°.) The circle of radius $CK = bb'' = bC'$, is the horizontal, and the tangents to it, as Kh, are the vertical projection of the *vertical cylinder of revolution*, from which the face-joints, mr, np, are cut, as at $e'e$, $e''g$, etc., by planes $c'f$, $c''h$, etc., perpendicular to V, and oblique to the axis of the cylinder. These planes will therefore cut the cylinder in ellipses whose semi-transverse axes are $c'f$, $c''h$, etc.; and whose semi-conjugate axes are each equal to the radius, CK, of the cylinder.

These ellipses, being thus known by their axes, each may be shown *first* by revolving it into, or parallel to a plane of projection; or, *second*, all may be shown in one figure constructed on a common conjugate axis.

First Method. The plane of np, may be revolved to the right about the axis; $sb''' — np$, parallel to the vertical plane, when, after revolution, $b'''N'$ will appear at nN'''; kM' at jj'', and sP' at pp''; all perpendicular to np. Then $p''j''N'''$ will be the *true form of the elliptic arc*, $N'P' — np$, forming a face joint. Others may be similarly found.

Second Method. Again, in Fig. 25, lay off from K', on $K'H$ produced, a distance = KC, from Fig. 22, giving a point O. Then on a perpendicular to $K'O$, at O, lay off, from O, distances to the left, equal to the semi-transverse axes, CK (of the circular right section in the plane CK) $c'f$, $c''h$, etc., and construct quarter ellipses on $K'O$ and these several semi-transverse axes. Finally, since CD, $c'e'$, $c''e''$, etc., are the perpendicular distances of the inner extremities, D, e', e'', etc., of the joints from the conjugate axes, perpendicular to the paper at C, c', c'', etc., lay off these distances from O on the semi-transverse axes in Fig. 25, giving points from which draw ordinates, parallel to $K'O$, which will give D, e', e'', e''', etc., corresponding

to D, e', e'' e''', etc., in Fig. 22. The elements of the intrados being parallel to the conjugate axes, at C, c', etc., of the ellipses, Fig. 22; aDc, $de'f$, $ge''i$, etc., Fig. 25, are bevels giving the true forms and positions of the face-joints Dd, $e'e$, $e''g$, etc., Fig. 22, relative to the elements of the intrados.

Returning to the joint np; pp'', jj'', and nN''', being revolved positions of lines parallel to the axis of the arch, if we lay off, from p'', j'' and N'', on these lines produced, distances equal to $N'N''$, $M'M''$ and $P'P''$, we shall have the true form of the face-joint, np — $N''P''$, and of the pattern No. 5, of the radial bed $N'N''M'M''$. The pattern of the radial bed at mr may be similarly found.

4°. The soffit mn. The patterns of this, and of all the other like surfaces, are shown on the development, Fig. 28, of the entire intrados, ABD, of the arch. The length of this development, $= 2AC$, Fig. 28, $= 2Amn$B, Fig. 22. Projecting over GI from $M'M''$, and HJ from $N'N''$, and proceeding likewise for the other stones, we get GHIJ as the pattern No. 6, of the surface mn — $M'M''N'N''$, for example.

5°. The patterns, Nos. 8 and 9, of the two faces of the stones will be found at once by developing the concave, and the convex cylindrical faces, $A'C'D'$, and $A''C''D''$, of the arch.

6°. Having now the patterns of all the faces of the stone mn pq, their relative position may be determined by the square as shown at S; a bevel No. 10, set to the angle npq, and an arch square, as VY, No. 11.

7°. Both plane beds being first finished, the templet vw, No. 13, might replace No. 11.

III. *Application.* This, so far as not already evident from the description of the directing instruments, would be as follows.

Taking the stone just considered, the radial bed on np, would first be wrought by No. 1, the straight edge, and the pattern No. 6. Thence the top could be wrought by No. 10, and the pattern, No. 3; and the soffit, by No. 11, and the pattern, No. 6. Also No. 2, held in planes parallel to the face qr, will give elements of the cylindrical faces of the stone in their true relation to the top pq. Or, a frame TT', No. 12, whose parallel bars TT and $T'T'$ are curved to the radius of the face, and held in planes parallel to the top, pq, would give circular lines of the face, between which No. 1, held perpendicularly to

TT, would test the proper cutting away of the intermediate rough stone.

EXAMPLES. — 1°. Substitute for the circular wall, shown in the plan, a straight wall, with a sloping face, as shown in Fig. 24, and make the isometrical drawing of any one of the stones; as indicated in Fig. 26 when the batter is on the *front*.

Ex. 2°. Make the isometric drawing, Fig. 23, of a stone from the cylindrical wall.

Ex. 3°. Construct any of the arches already described, on Pls. II. and III. on a large scale; with an isometrical drawing of any one of the stones; also a development of all the faces of one stone into the plane of the paper.

Ex. 4°. Do. for Pl. III., Fig. 27.

Ex. 5°. Do. for Pl. III., Fig. 29, supplying a plan, and batter at one end.

Ex. 6°. In Prob. IV. let the face of the arch be in a *vertical plane*, but oblique to the axis of the arch.

Ex. 7°. Let the arch, either segmental or full centred, be in a recess, with divergent sides, as in the plate band, Prob. III.; and therefore conical above the springing lines.

PROBLEM V.

A semi-cylindrical arch, connecting a larger similar gallery, perpendicular to it, on the same springing plane; with an inclosure which terminates the arch by a sloping skew-face.

I. *The projections.* Pl. IV., Figs. 30–32.

1°. Let the following be the given dimensions.

Skew (24) of the oblique wall PQC$_4$,		$= 18°$.
Batter " " " " "		$= \frac{3}{10}$.
Radius, O′A′, of intrados of arch		$= 3'$.
" O′a' " extrados " "		$= 4' : 6''$.
" O′K, " intrados of gallery		$= 6' : 6''$.
	Least thickness of wall	$= 2' : 3''$.
	Greatest " " "	$= 5' : 4''$.

Let the springing plane be taken as the plane **H**, and let the given thicknesses of the wall be in it; and let the plane **V** be perpendicular to the axis, O″O — O′, of the required arch. Let there be five voussoirs, dividing the section A′E′B′, of the intrados equally, and let them be completed by horizontal and vertical planes, as C′D′ and G′C′, through the outer extremities, D′ and G′, of the radial beds.

Then, with a scale of not less than $\frac{1}{24}$, $= 2'$ to $1''$, in order to be more easily accurate, the given dimensions can be drawn, as shown, where JR is the horizontal trace of the vertical side of the wall.

JR — KR′ is one springing line of the gallery, all of whose

elements are therefore parallel to JR, in front of the vertical plane JR, and above the plane H, of the springing lines, JR — KR′; and AA″ — A′, and BB″ — B′, of the arch.

2°. *Declivity of the plane face of the arch.* — PQ, at 18° with JR, is the horizontal trace of the plane of this face. Its batter, $\frac{3}{10}$ (3 to 10), is perpendicular to PQ. Hence, assume bL, perpendicular to PQ, revolve it about a vertical axis at b, to bL″, parallel to V, make $b'p' = 10$, from any convenient scale of equal parts, and p'L′ = 3, from the same scale; and L′b' will be the revolved position of the line of declivity, bL, showing the real slope of the plane arch face.

From this batter we find next, for convenience in projecting points, the slope, taken in vertical planes parallel to the axis, O″O — O′. Thus LI, parallel to PQ, is the horizontal projection of a horizontal line in the plane face of the arch. Note I, its intersection with the vertical plane bb'', revolve I to I″, project it to I′, on p'L′, and I′b' is the vertical projection of Ib, and is the declivity of the plane face of the arch, in the vertical plane bb''.

3°. *Horizontal projection of the plane face.* — Draw horizontals, through all the points of the face, as C′C_1 through C′ and D′; produce them to meet I′b', as at C_2; then, for instance, project C_2 at C_3, and revolve it to C_4; then C_4 C, parallel to PQ, will be the horizontal projection of C′C_1, and will intersect the projecting lines, from C′ and D′, at C and D, the horizontal projections of C′ and D′. Find other points similarly, or —

Otherwise: project k' at k'' and revolve it to k, when kC will coincide with C_4C, whence, as before, etc.

Again: as C_1C_2 is the true distance of C′ and D′ in front of the vertical plane on PQ, and in the direction of the axis O″O — O′, make iD = jC = C_1C_2, and we have C and D as before.

The horizontal projections of the radial joints all meet at O.

The semi-elliptic face line, AEB, has AB and 2OS for a pair of conjugate diameters; hence it is tangent to AA″ and BB″ at A and B, and at S has a tangent parallel to AB.

50. The last of the three constructions of CD, etc., just given, namely, by the method of *compass transference of known distances is advantageous*, in avoiding numerous lines of construction, as all from C_2 to C; yet for the same reason, *disadvantageous*, in not preserving upon the paper such traces of the construction as would enable any one to recall it from the drawing alone.

4°. *Horizontal projection of the cylindrical face.* — JKJ′ is a profile plane, which contains a semicircular right section of the cylindrical gallery. Revolving this plane about a vertical axis at J, the centre of that section will appear as at O_1 by making KO_1 equal to the given internal radius of the gallery; and Kc_2, with O_1 as a centre, will be the revolved position of the section. Thence the horizontal projection of any points of the cylindrical face of the arch can be found as before.

Thus, produce D′C′ to c_1, and then either make $cC'' = c_1c_2$, the true distance of C″C′ in front of the vertical plane on JR; or, by showing the counter revolution, etc., project c_2 on JR, at c_3, not shown, counter revolve c_3 to c_4, not shown, on KJ produced, whence project it by a line parallel to JR, till it meets C′C″, giving C″. In like manner all points of the cylindrical face may be found.

The radial joints D″E″, etc., of this face are arcs of ellipses, being sections of the cylindrical intrados of the gallery by the planes OO′D′, etc., which cut it obliquely.

Opposite joints, symmetrical with OO″, as D″E″ and d_1e_1, form parts of one ellipse, $D''O''d_1$, in horizontal projection, since $d_1e_1 - d'_1e'_1$ is exactly over that part of the ellipse D″O″ — D′O′ whose vertical projection is on D′O′ produced.

All the lines of the cylindrical face are invisible, and hence dotted, except such top and lateral edges, as C″D″ and C″G″.

II. *The directing Instruments.* — We may either show together the patterns of *like faces of all* the stones, or the patterns of *all the faces of one stone.*

Adopting the latter method as clearer, while illustrating all the operations required by the former, let the stone C′D′F′ be chosen for detailed representation.

Development of the stone C′D′F′. — A right section of this stone is the polygon C′D′E′F′G′, which will develop in a straight line as A_1B_1, Fig. 31.

Then, supposing the top face to be the plane of development, and that the faces to the right of D′, around to G′, are developed to the right of the edge D″D — D′, while the face C′G′ revolves to the left about the edge C″C — C′ into the plane C′D′; we shall make, in Fig. 31, A_1c; cd; de; ef; and fB_1 equal respectively to G′C′; C′D′; D′E′; E′F′, and F′G′ in Fig. 30. At these points draw lines perpendicular to A_1B_1,

for the indefinite developments of those edges of the voussoir which are parallel to the axis, O″O — O′.

To develope the edges of the plane end of the stone. — Lay off the perpendicular distances of their extremities from some plane of right section, as JR. Thus $A_1g_1 = cG$; $cC_1 = cC$, etc., the second term of each equality being taken from the plan in Fig. 30. The *curved edge*, EF —E′F′, is developed by taking one or more intermediate points, as HH′, and making eh, and hH_1, Fig. 31, respectively equal to E′H′, and $H h_1$, in Fig. 30. Also, for additional accuracy, make $dO_2 = D'O'$, and $O_2O_1 = O''O$, then O_1, Fig. 31, will be a point of D_1E_1 produced. Likewise, make B_1o_2 and o_2o_1, Fig. 31, $= G'O'$, and O″O, Fig. 30, and o_1 will be a point on G_1F_1 produced.

To develope the edges of the cylindrical face of the same voussoir. — As before in Figs. 31 and 30, respectively, $A_1g_2 = cG''$; $cC_2 = cC''$; $Nn = Nn$; $H_2h = H''h_1$, etc. Also, as C″G″ — C′G′ is an arc of the right section of the gallery, C_2g_2, Fig. 31, is drawn with a radius $= O_1K$, Fig. 30, and laid off on the perpendicular at the middle of the chord C_2g_2. The elliptic arc D″E″O″ is developed in its real size at $D_2E_2O_2$; and as every edge of the cylindrical face is curved, except C_2D_2, they will be curved in the development, and must be there found by intermediate points, as, at Mm, $= Mm$, Fig. 30.

To develope the plane face of the voussoir. — Revolve it about CD — C′D′ into the plane of the top. The true lengths of Gg, Ff, and Ee, Fig. 30, found at $k'r$, $k's$, and $k't$, on the line of declivity $L'b'$, will then be laid off on gG, f_1F, and e_1E, perpendicular to C_1D_1, Fig. 31; where g, f_1 and e_1 are transferred, as shown, from CD, Fig. 30, to its equal, C_1D_1, Fig. 31. The true size of the radius EO — E′O′ is found by revolving it about OO″ into the plane H, at $e'''O$. That of FO — F′O′ is, likewise, $f'''O$; then arcs from E and F, Fig. 31, as centres, with these radii, respectively, will give O_3, the development of the centre of the ellipse of which the face line, EF, is a part, and the point to which GF and D_1E radiate.

To develope the cylindrical face of the voussoir. — Here Fig. 32, Ff, for example, is the true length, taken from c_2f_2, Fig. 30, of the circular arc whose vertical projection is $F'f'$. Other points, being similarly found and joined, give the pattern, which must be flexible, of the cylindrical face. When flat, all its edges are curved except CD and GG. The developed faces of

the voussoir are the patterns of those faces. Those of the other voussoirs would be similarly found.

III. *Application.*—Select a stone whose right section will circumscribe C′D′F′G′, and whose length shall not be less than Ee'', and work the top by No. 1, the straight edge (7) and No. 3. Make the left side square with the top by No. 2, and finish it by its pattern. No. 4, with the pattern of the bed at D′E′, will complete that face. The arch square, No. 5, with the pattern of the intrados, will give the surface EE″FF″; and in like manner the whole convex surface can be wrought. All the edges of both *ends* of the stone being thus given, their patterns, with the straight edge, will serve to complete them.

Otherwise: the plane and cylindrical *ends* can be wrought next after the top, by bevels Nos. 6 and 7, respectively.

This problem includes the following simpler cases, added as examples.

EXAMPLES. — 1°. Let both ends be plane; one, vertical on PQ; the other vertical on JR.

Ex. 2°. Let the cylindrical end be replaced by a plane one, having JR for its horizontal trace, and a batter of $\frac{1}{3}$.

Ex. 3°. Construct a rampant (ascending) arch covering a straight flight of steps.

Groined, and Cloistered Arches.

51. Both of these kinds of arches are compound, being formed by the intersection of two single arches, each of which

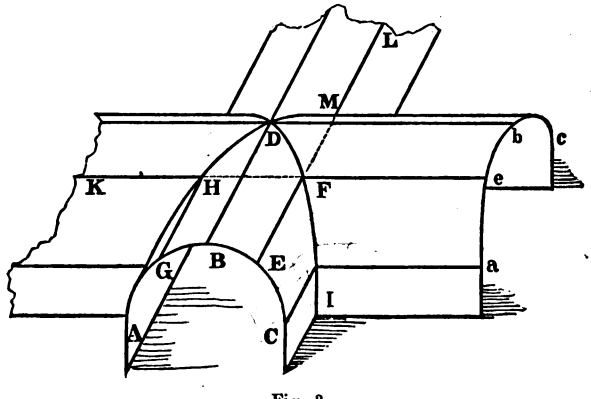


Fig. 3.

is usually cylindrical. They are distinguished from each other as follows: —

In the *groined arch*, Fig. 3, that part of each cylinder is real which is *exterior* to the other. Thus EF, LM, *e*F, and KH are real portions of elements.

In the *cloistered arch*, that part of each cylinder is real which is within the other. Thus in Fig. 3, HF and FM would be the real portions of the elements.

The *groined arch* therefore naturally covers the quadrangular space at which two arched open passages, ABC and *abc*, intersect; while the *cloistered arch* forms the doubly arched cover, or quadrangular dome, of a quadrangular enclosed room, or cell.

THEOREM I.

Having two cylinders of revolution, whose axes intersect, the projection of their intersection, upon the plane of their axes, is a hyperbola.

See Pl. IV., Fig. 30, where $O_2O - O'$ is the axis of the cylinder A'S'B', and a parallel to JR through O_2, where $O''O_2 = O_1K$, is the axis of the cylinder all of whose elements are parallel to the one JR.

A''S''B'' is the horizontal projection of the intersection of these cylinders, that is, upon the plane of their axes, and it will be shown to be a hyperbola by proving that $(O_2h'')^2 - (H''h'')^2 = (O_2S'')^2$; where H'' is any point on A''S''B'' and H''h'' a perpendicular (ordinate) to the axis O_2O.

Suppose $(O_2h'')^2 - (H''h'')^2 = (O_2S'')^2$. (1)

Now $(H''h'')^2 = (H'h''')^2 = S'h''' \times sh'''$; where $S's = A'B'$.

And $S'h''' \times sh''' = S'h''' \times (S'O' + O'h''')$

$= S'h''' \times (S_2s_2 + H_2h_2)$.

But $(S_2s_2)^2 = (O_1S_2)^2 - (O_1s_2)^2$.

And $(H_2h_2)^2 = (O_1S_2)^2 - (O_1h_2)^2$.

Whence, since $O_1s_2 = O_2S''$, and $O_1h_2 = O_2h''$

$(S_2s_2)^2 - (H_2h_2)^2 = (O_2h'')^2 - (O_2S'')^2$.

But also $(S_2s_2)^2 - (H_2h_2)^2 = (O'H')^2 - (O'h''')^2 = (H'h''')^2 = (H''h'')^2$.

Hence $(O_2h'')^2 - (O_2S'')^2 = (H''h'')^2$.

Or, $(O_2h'')^2 - (H''h'')^2 = (O_2S'')^2$.

Thus equation (1) is proved; and, calling $O_2h'' = x$; and

$H''h'' = y$; and $O_2S'' = a$; we have $x^2 - y^2 = a^2$; which is the equation of an equilateral *hyperbola*, that is, one whose axes are equal. Thus the theorem is proved.

52. *An important consequence* of the last theorem is, that when the cylinders become equal, the vertices of the curve, of which S'' is one, unite at O_2. Now when the two vertices of a hyperbola coincide, the curve reduces to the special case of *two intersecting straight lines.*

But the actual intersections of two cylinders (whose axes are not parallel) are curves. Hence if any projection of these curves is *straight*, they are *plane curves*, and hence *ellipses.*

53. A little consideration of the properties of hyperbolas will sufficiently show, what there is not room here to strictly prove: 1*st*, that a change in the angle between the axes of the cylinders would only cause the curve $A''S''B''$ to become a hyperbola referred to the new axes as conjugate diameters; and 2*d*, that the substitution of elliptic for circular cylinders would only yield a *general*, in place of the equilateral form of the hyperbola. The conclusion of (52) is therefore true of all cylinders whose diameters, measured in a plane perpendicular to that of their axes, are equal.

54. Cylinders, situated as just described, will have a pair of common tangent planes parallel to that of their axes. This fact, added to the last two articles, affords several statements of what is really the same proposition, each statement being appropriate to certain given conditions. Thus —

1°. If two ellipses intersect in a common semi-axis, of the same kind for each, their other axes being in the same plane P (thus if two ellipses whose transverse axes bisect each other in **H** have a common vertical semi-conjugate axis), lines joining points, which are on the two ellipses, and which are at equal distances from P, will be parallel, and will therefore form a pair of cylinders, of which these ellipses will be the intersections.

2°. If two cylinders intersect in one plane curve, as an ellipse, there will be a second branch of the intersection, which will also be plane.

3°. If two cylinders whose axes are in the same plane P, also have two common tangent planes, parallel to P, they will intersect in two plane curves, which will cross each other at

the intersections of the elements of contact of the tangent planes.

PROBLEM VI.

The oblique groined arch.

55. *Design.* — Suppose that, in the collecting system of certain water-works, supplied by several ponds, two conduits, each covered by semi-cylindrical arches of *nine* feet span, unite at an angle of 67° : 30′ and discharge into one of *twelve* feet span, covered by a semi-elliptic arch, having the same rise and springing plane as the former ones.

56. We may note in passing that, supposing the water to be *four* feet deep in each of the nine-foot conduits, the sum of their water sections is 72 square feet. Then, in the large conduit, if the water be but four feet deep, this conduit should be 18 feet wide. Or, if but 12 feet wide, as in the problem, its floor should be sunk so that the water in it should be *six* feet deep; or else its declivity should be increased to give such a velocity to the water in it that its section of 48 square feet would transmit as much water per minute as passes the sections of 36 square feet each, of the two nine-foot conduits.

I. *The Projections.* — Three planes of projection are used: the horizontal plane, **H**, containing the springing lines, H*f* and YN, YM and Y″M″, of the two smaller conduits, and HH'_1 and $H''H'_2$, of the larger one; a vertical plane **V**, whose ground line is P′Q′, and which is perpendicular to the axis OX — X′ of one of the smaller conduits; and a vertical plane $\mathbf{V}_1$, whose ground line is $H'_1H'_2$, and which is perpendicular to the axis O*m* — O'_1 of the larger conduit.

The lines in the horizontal plane can be first laid down from the given dimensions and axes, OX and O*m*, of the arches; giving the four springing points H, Y, Y″, H″, of the groin. Then OY and OY″ are the horizontal projections of the quarter ellipses (52), in which the arch whose axis is OR intersects the *right* hand half of the one whose axis is OX. Likewise OH and OH″ are the projections of the intersections of the arch whose axis is O*m* with the *left* hand half of the one whose axis is OX.

Next make the elevation on P′Q′, making the spandril stones 40″ thick, the thickness O′I′ at the crown 3 ft., and the radius,

$q'I'$, of the extrados, 9 ft. The radial joints from X' divide the intrados into five equal parts.

The elevation on $H_1'H_2'$ is now made as follows: —

1°. *The face line* $H_1'O''H_2'$. — Any point, as a', is the vertical projection of the element $A''Aa$, or of any point of it. Hence a' is horizontally projected upon the groin curves at A and A''. The projections of these points on V_1 will then be at a_1' and a'', at heights above $H_1'H_2'$ equal to that of a' above PQ.

2°. *Other points of the elevation on* V_1. — As the extrados is seldom a finished surface, we need not construct the groin curves of the extrados, but may proceed as follows, to find extremities of radial joint lines; these joints being *normal to the intrados* in both elevations.

To find f_1' for example. Draw the tangent $g'l'$ perpendicular to $X'g'$, project l' at l in the vertical plane OH of the groin, when Gl will be the horizontal projection of $g'l'$, considered as the tangent to the groin HO at Gg'. Then projecting G at g_1' and l at l_1', gives $g_1'l_1'$ as the new vertical projection of this tangent, and mg_1', perpendicular to it, as the like projection of the joint in the face of the main arch, corresponding to $X'g'$ on that of the arch $H'O'Y'$. Finally f_1' is the intersection of the joint mg_1' with $e_1'f_1'$ parallel to the ground line $H_1'H_2'$, and at a height from it $e_1'P'' = e'P'$.

In like manner, other points may be found, as may be seen at j_1', extremity of an auxiliary joint $X'j' - m_1j_1'$.

3°. The curve $b''I'''c_1'$, right section of the extrados of the elliptic arch is not, as might be supposed, an ellipse, derived from $b'j'$ as $O''H_1'$ was from OH; since the two conditions of normal joints, and equal heights of like joints on both elevations prevent this.

That is, i_1'', for example, determined from i' by projecting i' upon DO as the straight horizontal projection of the outer groin, and thence to i_1'', as g_1' was found from g' would not coincide with i_1'' of the figure, found on the given normal mg_1' and at the same height as i'; since each determination is complete in itself and hence independent of the other.

Points, as B, C, etc., of the *horizontal projection* of the outer groin, are at the intersection of projecting lines from b' and b_1'; c' and c_1', etc.; and are not in a straight line with O; thus showing that the outer groin is not, in space, a plane curve.

4°. *The Projections of a Stone.* — The stones of a groined arch in jointed masonry are partly in each arch. Hence, taking the most irregular stone as an example, it is that whose section in the front elevation is $a'b'c'd'e'f'g'$. The side elevation of the end in the elliptic arch is $a_1'b_1'd_1'e_1'g_1'$, and its plan is limited by the figure A*ad*Dd_1a_1.

II. *The Directing Instruments.* — Passing by Nos. 1 and 2, the straight-edge and square, which are of constant use (7), there are the following: —

No. 3, = the pattern $a'b'd' \ldots . g'$ of the end in the vertical plane at *ad*.

No. 4, = the pattern, *dc*Cc_1d_1, of the plane portion of the top.

No. 5 = the pattern F*fdd*$_1f_1$ of the plane portion of the under side.

Fig. 34 shows, together, the patterns, Nos. 6, 7, 8, and 9, of the principal lateral faces within the circular arch. There, $ab = a'b'$; bc = the arc $b'c'$; ag = the arc $a'g'$; and $fg = f'g'$. Also C*c*, B*b*, A*a* ..... F*f* = C*c*, B*b* ..... etc., in the plan, Fig. 33. Thus No. 6 is the pattern of the plane radial bed AB*ab* — $a'b'$; No. 7, of the cylindrical extrados BC*bc* — $b'c'$; No. 8, of the cylindrical intrados, AG*ag* — $a'g'$; and No. 9, of the plane radial bed, FG*fg* — $f'g'$.

Detailed description of the analogous patterns, Nos. 10, 11, 12, and 13, Fig. 35, of the corresponding lateral faces of that part of the stone lying in the elliptic arch, is unnecessary, as the construction of Fig. 35 is entirely similar to that of Fig. 34.

Thus, a_1p_1, Fig. 35 = $a_1'p_1'$ from the elliptical elevation; and Aa_1 Pp_1, etc., Fig. 35, = Aa_1 Pp_1, etc., from the plan; hence AGa_1g_1 is the pattern of the soffit AGa_1g_1 — $a_1'g_1'$.

As a further aid to accuracy, there should be one or more right section bevels, as Nos. 14 and 15, to test the relative position of the lateral faces. No. 16 is the pattern of the end in the plane a_1d_1.

III. *Application.* — Choosing a block of the thickness *hk*, and in the plan of which A*add*$_1a_1$ can be inscribed, first work the end in the plane *ad*, and complete it by the pattern No. 3. From this, by means of No. 2, the square, determine the direction of all the lateral faces, including the upper and under ones; and the back, whose horizontal projection is D*d* and Dd_1; and finish them by their patterns, Nos. 4–9, and the arch-square, No. 14.

This done, the three edges, c_1d_1, f_1d_1, and $d_1 - e_1'd_1'$ of the end in the plane a_1d_1 will be known, whence this end can be wrought square with the top and bottom plane surfaces, and completed by its pattern, No. 16. Thence the remaining lateral surfaces within the elliptic arch can be directed by the square, and completed by their patterns, Nos. 10–13, and No. 15.

EXAMPLES. — 1°. Let the cylinders (having a common springing plane, and equal heights in every case) be equal.

Ex. 2°. Let them be at right angles.

Ex. 3°. Construct the patterns for the key-stone (which will extend in one piece from O, on all the cylinders).

Ex. 4°. The arches being at right angles, let the circular cylinder be real between OH and OH″, and the elliptic one, between OH and OY, forming a cloistered arch.

Ex. 5°. In Ex. 4°, construct the patterns for the groin stone corresponding to the one described in the groined arch (Aa will be real from A *forward*, and Aa_1 will be real from A to the *right*).

Ex. 6°. In Exs. 2° and 4°, let the extrados be finished as in Pl. IV., Fig. 30.

PROBLEM VII.

The groined and cloistered, or elbow arch.

57. This problem is designed to illustrate very compactly, that is, without repetition of similar parts, *first*, the difference between a *groined* and a *cloistered* arch; and, *second*, the mode of proceeding when it is determined that the intersection of the two extrados, that is, the *outer*, as well as the inner groin curve, shall be a vertical ellipse.

I. *The Projections.* — A gallery, Pl. VI., Fig. 47, whose width A′B′ is 8 ft., intersects one of 10 ft. in width, each of them ending at its intersection, ADB, with the other. These galleries are covered by arches of equal rise, C′D′ = C″D″, and whose axes, C′D and C″D, are in the same plane. They therefore intersect in a vertical ellipse (52), whose horizontal projection is ADB. The radius, E′C$_1$, of the given extrados is 5′ : 3″.

The portion A′C′DC″A″ thus forms one quarter of a right-angled *groined* arch; while B′C′DC″B″ forms one quarter of a right-angled *cloistered* arch. The construction will mostly be obvious on inspection, by comparison with Pl. IV., Fig. 33.

A stone of the cloister. — MPRr — M′P′ is its circular intrados; OPRQ — O′P′ is a radial joint; NOQq — N′O′ is its circular extrados; MNqr — M′N′ is another radial joint, and MO —

M′N′O′P′ is its vertical plane end. The like surfaces in the elliptic portion are obvious by reading the drawing.

The proposed *elliptical outer groin* may coincide with the inner one AB, in plan. That is, both may be in the same vertical plane. Then, taking the joint at $a'c'$ for illustration, c' will be horizontally projected at n' instead of at c; and n will thence be projected at n'', at the same height as c'; likewise F, at f, on BA produced, and at f''; and so on for other points. Then $E''n''f''$ will be an ellipse, it being the right section of the cylinder whose oblique section is the vertical ellipse Dnf. Producing the normal joint $a''c''$ to the new extrados $E''n''f''$, we find its new outer extremity o''; and the other vertical projection, $o'o'''$, at the same height as o'', of the element of the new extrados, containing o''. But $o'''o'$ must be limited by the plane, $C'c'$, of the joint $a'c'$, as at o', whose horizontal projection is o, on the horizontal projection, os, of the element at o''.

Considering then the stone *below* the joint $a'c'$, $a''c''$, its radial surfaces are $si''ao$; oan, where $on - o'c' - o''n''$ is an elliptic arc; and $ui''an$.

This design is more complex than the usual one, but gives a greater increase of radial thickness toward the springing of the elliptic arch. Indeed, unless the radius C_1E' exceeds a certain limit, relative to $C'D'$ and $D'E'$, the joint, or normal $e''F''$ will be less than $D''E''$, which is quite undesirable, relative to stability. The joints of the intrados, being the most important, are inked full; that is, as if seen from below; as it is often convenient to do (8).

The outer groin may be elliptical, found, like the inner one, from given extrados, if the consequent deviation of the derived face joints from a normal direction be only very slight.

II. *The Directing Instruments.* — These can be constructed as in the last problem.

III. *Application.* — See also the last problem.

EXAMPLES. — 1°. Let both arches be circular.

2°. Let both be elliptical.

3°. Let their axes intersect at any other than a right angle.

4°. Turn the figure right for left, and ink the plan as seen from *above*.

5°. Construct the patterns for the key-stone of the groin; and of any other stone, both in the groined and in the cloistered angle.

6°. *Assume* an elliptic extrados of suitable proportions on the elliptic arch, and *find* the corresponding curve joining the corresponding extremities of the normal joints of the circular arch.

Conical or Trumpet Arches.

58. When two walls, whether having vertical or sloping faces, meet, so as to enclose any angle, it is sometimes desirable to connect them at or near the top by self-supporting masonry, which may, for example, afford additional area for standing or passage. The original ground room enclosed by the supposed angle at which the walls meet is not encroached upon by the arched connection of the walls, since the latter is self-supporting from the walls.

Such arch work, projecting from one or more wall faces, or piercing them, and generally with a conical intrados, has been called a trumpet.

PROBLEM VIII.

A trumpet in the angle between two retaining walls.

I. *The Projections.* — 1°. *Description.* Let AC and BC, Pl. V., Fig. 36, be the horizontal traces of two walls, enclosing a right angle; and whose faces, C*A*d*F*, and C*B*e*F*, have the same slope or batter of 2 to 5.

Let FDE be the level top of a quadrantal platform, with a radius of 6′ : 10″; and forming a quarter of the base of an inverted oblique cone, whose axis coincides with FC — F′C′, the line of intersection of the faces of the walls; and a portion of whose convex surface forms the front face, ADEBJ — A′D′E′ B′J′, partly broken away on the right, of the platform.

This platform is then supported by a trumpet, whose surface, AJBC — A′J′B′C′, is a segment of a cone of revolution whose axis HC — C′ is perpendicular to the plane **V**, and in the plane **H**. From this general description, we proceed to the details of the construction.

2°. *Outlines of the walls and platform.* — Having laid down the traces, AC and BC, of the walls, CC′, the horizontal projection of their intersection will bisect ACB, because the walls have the same slope. Next draw C*m*, and on it lay off from C, *five* from any scale of equal parts; and *lx*, parallel to BC = *two* from the same scale; then, by the given conditions, C*x* will be the section of the face of the wall A*d*C′, made by the vertical plane, on BC, and revolved about the trace BC, into

H. Now lay off Cf = the height of the platform, and Cm, that of the wall, draw fF'' and mM', parallel to lx, project M' at C', and F'' at F, on CC', and draw $C'd$ and $C'e$ for the horizontal projections of the upper edges of the walls, and FD and FE, respectively parallel to them, and of the given dimensions for the intersections of the platform with the faces of the walls.

3°. *Conical front of the platform, and its vertex.* — Supposing it required for good appearance that the conical front face of the platform shall intersect the walls in their lines of declivity, draw DA and EB, perpendicular to AC and BC, as such intersections; then AKB, with C as its centre, will be the horizontal trace of this conical front: and as DA and EB are two of its elements, their intersection, V, will be the horizontal projection of its vertex.

The projections of VD and VE on the vertical plane BCM, are BCM and B; which after revolution, as in (2°) will appear at M'C and EB produced, which meet at the point indicated as V''. Then, making $C'V' = BV''$, we have V' the vertical projection of the vertex of the conical front of the platform.

4°. *Outlines and joints of the trumpet.* — Let the vertical semicircle, AHB — A'H'B', on the chord, AB, of the horizontal trace of the inverted cone, be the directrix of the cone of revolution forming the trumpet. Dividing A'H'B' conveniently into equal parts, here three, project i' and j', the points of division, at i and j, giving Ci — $C'i'$, etc., as elements, and joints, of the trumpet.

To find the *face line* of the trumpet, find where its elements meet the surface of the cone VV'. Since the vertex CC' of the trumpet cone is in the axis, VC — V'C', of the other one, this is easily done by assuming any element, as Ci — $C'i'$, of the trumpet and finding its trace, and that of the axis, VF — V'F', upon the plane D'E' of the upper base of the platform. These traces are N'N and F'F, giving FN as the trace of the plane of these lines upon the top of the platform. FN cuts the circumference, EID, of the platform at aa', giving aV — $a'V'$ for the element of the cone VV' in the same plane with Ci — $C'i'$, and hence intersecting the latter at bb', a point of the required face line.

The highest point. — This, JJ', is found by revolving the elements VI and CH of the two cones, and in their common vertical meridian plane VC, about any convenient vertical axis, as

the vertical trace, C′F′, of that plane, till they fall in, or parallel to the plane V. Each point revolved moves in a horizontal arc, at its proper height, as can be read from the figure, giving V‴K″I″ and C″H″ as the respective revolved elements. These intersect at J″; which, by counter revolution, gives JJ′, the required highest point.

To give a neater design to the top of the platform, the planes of the joints of the trumpet radiate from the axis of the cone VV′, giving the radial lines as *a*C for the joints of the platform. But to prevent the stones from coming to an edge along that axis, they abut against a conical faced stone whose upper base is *gkh*, of any convenient assumed radius, and whose intersection *ucv* — *u′c′v′* with the trumpet cone may be found as A*b*B — A′*b*′J′ was.

This stone may be built into the walls, as along the planes *Qgu* and *hv*P, to any extent desired for stability. Also the side stones, as A′D′*a′b′*, of the trumpet may be likewise built into the wall to avoid a thin edge along AC — A′C′. The small component of the reactions of the side stones tending to thrust the central one forward would be sufficiently resisted by the adhesion of the cement. Otherwise: the intermediate stones, one or more, could easily be supported by forming the stone *ghuv*, as indicated in Fig. 37, with a conical step as *s*.

II. *The Directing Instruments.* — These, taking the central stone for illustration, will consist of patterns of its four lateral faces, with a few bevels.

After Nos. 1 and 2, there will be No. 3, the pattern, *aknp*, of the top, No. 4, that of the two equal radial beds, and No. 5, that of the conical intrados; which are constructed as follows: No. 4 shows the full size, and real form of the radial bed *npqr*, which is supposed to be revolved about its horizontal upper edge *np*, till horizontal; when *r*, its lowest point, will be found at a distance s_1r_1, Fig. 42, from *np*, equal to the hypothenuse of a right angled triangle, of which *rs*, perpendicular to *np*, and the vertical distance of *r′* below *n′p′*, are the other sides. Also, $n_1s_1 = ns$. Finding t_1q_1 in like manner, we have the pattern No. 4, which will serve for both of the radial beds of this stone.

No. 5 is the development of the conical intrados, made by describing the arc $i_1j_1 = i'j'$, and with the radius C″H″, = the slant height to the circular directrix AB — A′H′B′, and laying

off the true lengths of the elements as $C_1b_1 = C''b''' =$ the true length of Cb — $C'b'$ revolved first at $C'b''$ into the vertical plane $CC'F'$ and thence to $C''b'''$.

Bevels, Nos. 6 and 7, set to the angles $s_1p_1q_1$ and $s_1n_1r_1$, and held in the plane $CC'F'$, will be useful in giving the positions of the two ends. The latter surfaces, being parts of oblique cones, can be developed only by the usual construction, in which the intersection of such a cone with a sphere whose centre is the vertex (VV') of the cone is found; the development of such intersection being a circle. But these developments are unnecessary.

Let *rs* now be considered as the horizontal trace of a vertical plane, perpendicular to the edge *np*, and cutting the adjacent radial bed in *sr*, and the top in a line also horizontally projected in *sr*. By revolving the former *sr* about the latter till horizontal, the true size of the diedral angle between the top, and the radial bed *npr*, will be found. A bevel, No. 8, set to this angle will be useful.

III. *Application.* — Having chosen a suitable block, in which the finished stone could be inscribed, first form the top, by No. 1, and by its pattern No. 3; next, the radial beds, directing their *position* by No. 8, and their *form* by No. 4; next the conical intrados, of which No. 5 is the pattern. All the *edges* of the two ends will thus be known, and as their radial edges are *elements* of the cones to which they belong, it is only necessary to transfer to *ap* and *bq*, from the drawings, points where elements meet those lines, and bring the front end *apbq* to its proper conical form by cutting away the stone till No. 1 will apply to it, at the corresponding points of division of *ap* and *bg*. Hence it is, that, as already said, the tediously found developments of these conical ends may be dispensed with.

Problem IX.

A trumpet arched door on a corner.

I. *The Projections.* 1°. *Outlines of the Plan.* — Pl. V., Fig. 40. The projections are here arranged partly with a view to the greatest compactness. Two walls of an enclosed space, and of the thickness, $AC = BD = 4' : 9''$, meet at right angles.

From a, the equal distances aA and aB, each $= 8' : 6''$ are set off as the external limits of the trumpet as seen in plan.

If CD, the width of the door were also given, it would determine the angle AfB at the vertex f of the conical surface of the trumpet. We here suppose AfB $= 90°$, which, with the previous data, makes CD $= 5' : 3\frac{3}{4}''$ very nearly. A vertical semicircle, AbB, which, revolved about AB till horizontal, gives AEFB, is taken as the base, or linear directrix of the trumpet. Then CdD is a vertical semicircular section of the trumpet, and also of a second conical zone CEFD whose vertex is c, and which serves to widen the approach, EFH, to the door. To avoid crowding the figure, CD is shown as a single line, which it might, in fact, be, in case of an opening having no gate, or of a thin iron gate; but in case of a gate of considerable thickness, the edge at CdD should be cut away giving a narrow cylindrical band, fitted to the gate top; or there should be a gate recess as in Pl. VI., Fig. 45.

2°. *The Elevation in general, and plans of the elements.* — The foregoing being the main features as seen in plan, the elevations are shown on two vertical planes; one having aB, and the other, A″B″ for its ground line. Of these, only the former is necessary for the purposes of the mason, showing as it does the true sizes of the lines in the external face of one of the walls; while the other, on the vertical plane at A″B″ is only useful as helping to give an idea of the structure, as seen in looking directly through the doorway, in the direction af.

The vertical planes Aa and Ba, being parallel to the respective opposite elements, Bf and Af, cut the trumpet cone ABf in equal parabolas; hence, to avoid a too great inequality in the sizes of the arch stones as seen in the exterior of the walls, divide the semicircle AfB, or its equal A″b″F″, into convenient unequal parts, the largest, B″l″ $=$ Bl' being laid off from the springing at B″ of the arch. Project l', or l''; h', or h''; etc., at l, h, etc., and through l, h, etc., draw flk, fg, etc., horizontal projections of elements of the trumpet. These at n, i, etc., pass to the conical zone CEFD, as at nm, and thence to the cylindrical band EGHF in parallel elements, as mo.

3°. *To find the parabola,* Aa, *in its own plane.*

1*st*. *Without the elevation on* A″B″. — The trumpet, being a cone of revolution, and its axis af horizontal, its elements fa, fg, etc., revolved about its axis, fx, and towards B, will come

to coincide with the extreme element fB produced, as at fa_1, fg_1, etc.; where aa_1, gg_1, etc., perpendicular to fx, are the horizontal projections of the arcs described by a, g, etc. Then a', g', etc., vertical projections of a, g, etc., extremities of elements of the trumpet, are at the intersections of the perpendiculars aa', gg', etc., to the ground line aB, with the arcs a_1a', g_1g', etc., all having x for their centre, and which being in the vertical plane on aB, are seen in their true size. (The arc k_1k', being confused with Bk', is not shown.)

2*d*. *With the use of the elevation on* A″B″. — Having h'', for example, vertical projection of h, draw $f''h''$, the vertical projection of the element fh, and project g upon it, at g''. Then g' is at a height gg', equal to that of g'' above A″B″; and in like manner other points of Ba', except a', can be found. As before, $aa' = aa_1$.

4°. *Determination of the radial beds.* — These, if the face joints, as g'I′, were made normal to the parabolas, of which Ba' is one, would be determined by these joints, with the element joints fg, etc., and hence could not also contain the axis fx of the trumpet, since g'I′, etc., if normal to Ba', do not intersect that axis. But if these radial beds do not contain the axis fx, which is also the axis of the cone cEF, and of the cylinder EGHF, their planes cannot cut the two latter surfaces in elements, and the stones of the trumpet would properly terminate in a vertical plane on CD, and be succeeded by others, radiating from the axis fcx, and covering the surfaces named between CD and GH.

We therefore choose beds radiating from the common axis fcx and extending from Aa and Ba to LH. The top edges of these beds, in the horizontal surfaces as I′K′ — I″K″ will then be parallel to fx; and the face joints g'I′ — g''I″, etc., will radiate from the point x, f'' in the plane Ba.

II. *The Directing Instruments.* — These, besides Nos. 1 and 2, (7) will consist of patterns of the surfaces of the stones, with such other bevels besides No. 2, as may be considered useful as checks.

Taking the stone $gkinmor$ — g'I′K′J′k' — I″K″J″$n''i''r''m''$, No. 3, the pattern of its back, is I″K″J″$r''m''$, which is in the vertical plane LH.

No. 4, is the pattern of its front, I′K′J′$k'g'$.

Nos. 5 and 6, are the patterns of the radial beds on f''J″,

and on $f''I''$. These are both shown as revolved about the axis fx of the trumpet, till they become horizontal. Thus $HI_2 = I''r''$; $g_1I_1 = g'I'$; Dg_1 shows the true length of $ig - i''g''$; etc.

No. 7, the pattern of the top, is a trapezoid of altitude $I''K''$ and bases equal to J_1J_2 and I_1I_2.

No. 8, is a pattern of the conical intrados, *gink*, found, if flexible, as in previous similar constructions by developing the cone whose vertex is f. But as it is only the elements of the intrados that must be found, it is enough to develop the pyramid whose edges coincide with these elements. Hence in Fig. 41, the chords Bl and lh are equal to the *chords* $B''l''$ and $l''h''$, Fig. 40 and *nikg* is the pattern required.

Flexible patterns 9 and 10 of *inm*, and *srom* can obviously be found. The vertical surface on $K'J'$, forming No. 11, is simply a rectangle, $= K'J' \times J_1J_2$.

Nos, 12, 13, and 14, will be bevels set to the respective angles $K''I''g''$, between the top and a radial bed; $K''J''k''$; and HFD, between a vertical side as $J'K'$ and the front, and taken in a horizontal plane.

Fig. 44, is an oblique projection of the stone just described, made intelligible by means of the like letters at like points.

III. *The Application.* — First, work the back by No. 1, and mark its form by No. 3. Second, all the lateral faces adjacent to the back are made square with it by No. 2. Also, the top and the vertical side on $J'K'$ are at right angles. Third, the *forms* of the faces just mentioned can then be marked by their patterns (5–7), 10, and 11; and the bevels, 12 and 13, can be used as checks on their *position*. Fourth, make the front square with the top, or at the angle HFD with the vertical side, holding No. 14 perpendicular to the edge $J'K'$.

Fifth, mark the elements, *gi* and *nk* by No. 9, whence *inm* can be wrought by No. 1, placed upon corresponding points of division of *in* and *ms* into equal parts.

The upper joints of the trumpet stones being parallel to fx, there will be some three-cornered stones adjacent to them in the wall, where the joints are parallel to AC and BD.

EXAMPLES. — 1°. Let the walls include any other than a right angle.

2°. Let their exterior be a vertical tangent cylinder from A to B.

3°. Let the cone be other than right angled at its vertex f.

4°. Let FH be increased till LL_1 shall be long enough to embrace the back ends of all the trumpet stones.

5°. Let LL_1 be a sloping wall.

6°. Let there be a batter to the exterior faces Aa and Ba, of the wall.

7°. Let there be no opening at CD, and describe with an oblique or isometric projection, the stone GECDFH—f, necessary to fill the opening, and admit the extension of the trumpet surface to its vertex f.

PROBLEM X.

An arched oblique descent.

I. *The Projections.*—Various conditions may give occasion for a structure of this kind. Thus it might lead from a side walk to an underground railway; or from a hydraulic canal to a turbine wheel pit; or it might cover an arched stairway leading to an arched gallery.

1°. *The perpendicular projections.*—ABCD, Pl. V, Fig. 43, is the horizontal projection of the section in the springing plane. A vertical plane on AC here makes an angle of 28° with a vertical plane on A_1 A perpendicular to AB. The line CD — D′ is one springing line of the intrados of a semi-cylindrical gallery, of radius MD = 11′ : 6″; the perpendicular length, AA_1, of the horizontal projection is 7 ft.; and AB and EF are respectively 15 ft. and 9 ft.

Two principal vertical planes of projection are used; V, that of the head, on AB, and one V_1 whose ground line is BD. The line AB is at a height, BB_1, = 3′ : 3″ above CD, and hence, strictly, the diameter AB of the vertical projection, AI′B, of the head should be a line A′B′ (not shown) parallel to AB and 3′ : 3″ above it. But to condense the figure, and because this *position* of AI′B is not essential, the vertical plane, V, of the head is revolved about its trace on the springing plane B_1DC, instead of about its horizontal trace.

2°. *The oblique projections.*—The projection of the arch upon the vertical plane, V_1, on BD, might be made in the usual way, by projecting lines perpendicular to that plane. But, as may be seen by trial, the result would be a much more complicated, but no more useful figure than the present projection on BD; which is an oblique projection, formed by projecting lines parallel to AB.

Thus, since the plane V is vertical, Bj'' perpendicular to BD, is its trace on the vertical plane V_1, and is also the oblique projection of the head, on V_1. Likewise, the quadrant DV, being

the revolved semi-right section of the gallery reached by the descent, M_1D is the oblique projection of its horizontal radius, MD, and similarly S, T, etc., are obliquely projected on V_1 at S_1, T_1, etc. Then making $S_1s_1 = Ss$; $T_1t_1 = Tt$, etc., Dt_1s_1 is an arc of the section of the gallery by the plane V_1, and Dj_2, where $j''j_2$ is parallel to B_1D, is the oblique projection of the cylindrical face of the descent. That is, $B_1j''j_2D$ is the oblique projection of the outlines of the descent.

Laying off the heights of the several points of the semicircular face above AB, from B_1 on B_1j'', and drawing lines parallel to B_1D and limited by Dj_2, through the points so found, we find the elements and edges parallel to them, of the descending arch. Thus, $B_1b'' = BB'$ by drawing $B'b$ parallel to BB_1 and the arc bb'' with centre B_1; then $b''i_2$ is parallel to B_1D, and all the other parallels to B_1D are similarly found, as may be seen by the figure.

3°. *The right section.* — This is in any plane perpendicular to the elements of the arch. To condense the figure (though at the expense of confusing it somewhat, having first sought to make it on the largest scale possible), assume XY, perpendicular to BD, and XI_1, perpendicular to B_1D, as the traces of such a plane. Then as usual, choose auxiliary planes parallel to the axis of the cylinder; here, vertical planes, parallel to V_1. Each of these will contain an element of the arch; and a line of the plane YXI_1, whose horizontal *trace* will be on XY and whose vertical *projection* will be parallel to XI_1. Thus the plane $I'PI''$, cuts from the arch the element at P''— P I'', and from the cutting plane YXI_1 the line whose horizontal trace is p'' (intersection of the horizontal traces XY, and Pp'') and whose vertical projection is pp', parallel to XI_1 through p the projection of p'' on XD. Having found, as above, $b''i_2$ also $p'q'$, by making $B_1q' = B_1q = BQ = PP'$, as shown; p', and i', the intersections of pp' with $q'p'$ and $b''i_2$ are two points of the oblique projection of the right section. Then making $p''P'' = pp'$ and $p''I'' = pi^1$, as shown by revolving p' and i^1 to p_1 and I_2, about p as a centre, and projecting p_1, and I_2 by the lines p_1P'' and I_2I'' to P'' and I'' on PP'', we have the position of the points p' and i' when revolved about XY into the horizontal plane.

All other points, both of the oblique projection and the revolved position of the right section, are similarly found, as is sufficiently indicated by the lettering of other points.

Making $Xe = Xe_1$, and drawing eY to Y, the intersection of XY with DC, the horizontal trace of the springing plane ; eY is the revolved position of the intersection of the plane, YXI, of right section, with the springing plane B_1DY.

4°. *Special Constructions.* — Ye is made to pass through F, in order to compare FO′E and FO‴E more nearly, though this is not necessary. Ye is thus placed by first assuming YXI_1 at pleasure, and finding the corresponding position of eY ; when, if this position does not contain F, draw a parallel to it that will, viz., eY as on the figure, which will meet DC at that position of Y whence the corresponding desired position of YX can be drawn.

The tangent to FO‴E, *parallel to* AB. Considering AB for a moment as a line in the revolved plane of right section and parallel to such a tangent, its horizontal trace would be y ; and its vertical trace B_2 would be found by making $XB_2 = XB$. Then making the height $Bb_3 = B_2b_2$, we get b_3y, its projection on V, and the parallel tangent at K′ gives K′K two projections of the point of contact of the required tangent from which K″ the point of contact on the revolved right section, of a tangent parallel to CD is found as in (3°).

II. *The directing Instruments.* — Taking the springing stone, FBB′I′R′, these will be as follows, besides Nos. 1 and 2. No. 3, the top, is a parallelogram of width B″I″, and length $b''i_2$. Then with centre B, and radius $b''i_2$ cut DM at i_3, and Bi_3 will be the position of b^2i_2 — B′, after revolving till horizontal, about AB as an axis, since i_2 is in that right section of the gallery whose horizontal projection is MD. Then i_3B and BP will be two sides of the parallelogram, No. 3.

No. 4, the pattern of the vertical *side* of the stone, is of perpendicular width = eB'', bottom length = B_1D, and top length = $b''i_2$. One end is the vertical line BB′, the other the arc Di_4 of DV (2°) corresponding to Di_2.

No. 5 is the pattern of the right section $eB''I''R''F$, used in one method of working the stone.

No. 6, = I′B′BFR′, is the pattern of the plane end.

No. 7, the pattern of the opposite cylindrical end, differs from No. 6, in that BB′ would be replaced by the development of the arc Di_4 ; and RR′, by that part of DV, corresponding to Dn_2, while the developed joint R′I′ would be curved, as

found by means of an intermediate point u, as in Prob. V., etc.

The remaining patterns require the construction of other developments. Making $F''R''N''E''$, Fig. 43 (taken on CD, only to bring the figure within the plate), equal to the right section $FR''N''E$, Fig. 42; note that the real distances, estimated on elements, from the right section, $e'o'f_1$, to the heads of the arch, are seen on the oblique projection. Then make $F''F$, Fig. 43, $= f_1B_1$, Fig. 42, and in like manner passing from one figure to the other, make $R''R = p'q'$; $O''O = o'o'''$; $N''N = n'N_1$, and $E''E = e'B_1$; and FRNE will be the development of the *face line*, $FO'E$, of the *plane end* of the arch.

Next, make FH, Fig. 43, $= B_1D$, Fig. 42 (*i. e.* $F''H = f_1D$), and likewise, $RR_1 = q'r_2$; $NN_1 = N_1n_2$, and $EG = B_1D$; and HR_1N_1G, will be the development of the face line of the cylindrical end of the descent. Then No. 8 $= FRR_1H$, the developed intrados of the stone considered.

Finally, the centre, OO_1, of the plane end, to which its joints radiate, is at the distance O_1B_1 from the centre $O_1(O'')$ of the right section; hence in Figs. 43 and 42, respectively, make O_1O_2, at this distance, O_1B_1 from the right section $F''E''$; make $RO_2 = R'O$; or $rO_2 = R''O''$, and $O_2RI = OR'I'$. Then, in the two figures, make $II_1 = b''i_2$, and $O_2o_2 = B_1D$, and $I_1R_1o_2$ will be the developed joint on the cylindrical end, corresponding to $R'I'O$, Fig. 42, on the plane end. Hence No. 9, $= RII_1R_1$, is the pattern of the radial bed on $R'I'$.

No. 10, = FBHD, Fig. 43, and similarly found, is the pattern of the springing surface whose horizontal projection is FBHD, Fig. 42.

Besides these patterns, bevels, Nos. 11 and 12, set to the angles $B''I''O''$ and $i_2b''B$, respectively, will be useful in one method of working the stone. Also No. 13, giving the angle, $BB''I''$, between the side and top of the stone, in a plane of right section.

III. *Application.* — 1°. *The method by squaring.* — Choose a stone on which a right section can be formed, exterior to the finished stone, by No. 1, and No. $5 = eB''I''R''F$. Next, make all the lateral surfaces square with this right section, by Nos. 1 and 2, and mark their edges by their patterns, Nos. 3, 4, 8, 9, and 10. This operation will give all the edges of both *ends*, which

can thus be formed by cutting away the rough stone on the ends down to them, applying No. 1, in a direction parallel to AB on both ends. This method is simple and accurate, but wasteful of the *stone* between the actual plane end and the exterior provisional right section, and of the *labor* of making the plane surface of this right section. It may, however, be employed in all cases, like many of the preceding, where the actual ends are curved, or oblique to the right section.

2°. *The method by oblique angled bevels.* — Choosing a block in which the finished stone can be inscribed, work first the vertical *side*, that being the largest, and mark its edges by No. 4. Second, work the top *square* with the former, *if* the arm of the square in the top be guided by a small *plane bevel*, laid in the top, and set to the angle i_3BF. Otherwise, use the level, No. 13, held perpendicularly to the top edge Bi (BD — $b''i_3$). Thence, finish the top, by No. 3. Likewise work the under side, and the radial bed on R′I′ — R″I″, the latter by No. 11, held perpendicular to the top edge, II_1, Fig. 43. Next proceed with the plane end, using No. 12 to give its position relative to the top. From the finished plane end, the lengths at all points being known from the side elevation, the remaining sides and the cylindrical end can be easily and accurately wrought.

EXAMPLES. — 1°. Make the side elevation in *perpendicular* projection.
2°. Let the arch ascend from the plane end to the gallery.
3°. Construct the indicated pattern, No. 7.
4°. Let the descent be direct, BD perpendicular to AB.
5°. Construct the patterns for the key-stone.
6°. To avoid confusion, take X to the right of B.

CLASS III.

Structures containing Warped Surfaces.

59. *Warped faced wing walls.* — Suppose that the inner faces, as $bm - b'm'$, Pl. I., Fig. 7, instead of being vertical, were sloping, but in such a way that the lowest lines of the fronts of the walls should be, as seen in plan, parallel to bm and np. Thus let them be as at $h'h''$. The *rate* of slope at mh, where the wall is highest, would then be less than at bh'', where the wall is lowest. The face of the wall would therefore be a warped surface, and would be a portion of a *hyperbolic paraboloid;* generated either by hh'', moving on bh'' and mh so as to remain horizontal; or by bh'', moving on bm and hh'', and parallel to the vertical plane on mn.

EXAMPLE. — Construct the case just described in a large figure, with an auxiliary elevation showing the face of one of the wing-walls; and take the joints to coincide with positions of hh'' and bh''.

PROBLEM XI.

The recessed Marseilles Gate.

I. *The Projections.* — 1°. *The problem is this.* Given a straight wall in which is a recess with diverging sides, and in the recess a round topped portal, closed by gates of like form; it is required to cover the top of the recess by a surface which shall be agreeable, easily constructed, and practicable in not interfering with the turning of the gate. Thus, having a vertical straight wall, Pl. VI., Fig. 45, bounded in thickness by the parallel planes AB and C''D; and in which is the passage EF, having a semicircular top, E'G'F', and covered by gates of like form, fitted to the recess EFHI — H'E'G'G''F'I'; it is required to cover the diverging recess or embrasure remaining between the vertical planes HI and AB, in the manner enunciated.

It will be agreeable that AH should be not less than HG, the width of the gate; and that the front top edge, AB—

A′K′B′, of the recess should be arched, in which case the vertical height, G″K′, and the radius, AO — A′O″, should be so adjusted that A′ and B′ shall not be lower than G″, the highest point of the gate.

So much being fixed, let the axis, OY — O″, of the arch, and the face lines, H′G″I′, of the gate recess, and A′K′B′, of the embrasure, be the three given directrices of a warped surface, generated by the motion of a straight line upon them. (Des. Geom. 251.)

One of these directrices, OY — O″, being straight, any desired elements of the proposed warped surface may readily be found by noting the points in which any plane containing OY — O″ cuts the other two directrices. Thus OO″A′ is a plane containing OY — O″, the point AA′ of the front face line, and cutting HI — H′G″I′ at L′L, giving ALY — A′L′O″ for an element of the warped surface.

But the limited directrices limit this warped surface by the elements AL — A′L′ and BM — B′M′, so as to still leave undetermined the surfaces projected in ALH and BMI.

On extending the warped surface just formed, by producing the directrix A′K′B′, it will generally intersect the sides, AH and BI, of the recess in curves, which would prevent the full opening of the gates.

We therefore proceed to complete the proposed top of the recess by means of warped surfaces having the two directrices, OY — O″, and H′G″I′, in common with the preceding warped surface, and for a new third directrix a curve through II′ and BB′, so formed as not to interfere with the full opening of the gate. This third directrix is conveniently shown, first, in its real form, by revolving the face, BI, to a position parallel to the plane V, when BB′ will appear at GB‴.

2°. *Determining conditions of the new third directrix.* These are: —

1*st.* That it should enclose I′G″, and be tangent to it at I′.

2*d.* That it should also contain the point B‴.

3*d.* That the new warped surface directed by it should be tangent to the preceding one along the common element, MB — M′B′, in order to avoid any visible edge of transition, or break, in passing this common limit of the two surfaces.

The last condition will be fulfilled if the two warped surfaces be made to touch each other at any three points of their

common element, YMB — O″M′B′. But this they evidently do at the two points, YO″, and MM′, since there the linear directrices are the same for both surfaces.

Let BB′ be the third point of YMB — O″M′B′, at which the two warped surfaces shall be tangent. For this purpose, they must there have a common tangent plane. Such a plane will be determined by two tangent lines at BB′, of which the most convenient are YMB — O″M′B′, which is tangent to itself; and B′T′, the tangent, at BB′, to the directrix A′K′B′. Now M′N′, parallel to B′T′, is the trace of this tangent plane on the plane HI; and N′, where it meets the intersection, I — I′N′, of the planes HI and IB, is one point of its trace on the plane IB. But BB′ is another point of the same trace, which is, therefore, in revolved position, N′B‴.

The third directrix of the new warped surface is therefore, as seen in the revolved position, a curve which shall be tangent at I′ to I′G″, and at B‴ to N′B‴.

3°. *Choice of curves.* — Preferring a natural, to an artificial curve for the directrix now determined, we may attempt an ellipse having either I′O″ produced for its transverse axis; or a line from I′, parallel to N′B‴ for a diameter. But in either case, unless its radius of curvature at I′ be not less than O″I′, it will intersect I′G″, and thus be impracticable. Hence the choice must generally lie between a curve of two centres composed of a part of I′G″, and an arc, tangent to it and to N′B″ at B‴; or a tangent line to I′G″ from B‴, with the portion of I′B‴ from I′ to the point of contact.

Preferring the former, draw B‴P, perpendicular to B′N′, and equal to I′O″; draw O″P, and a perpendicular to it at its middle point will meet B‴P produced at the centre of the required arc. But this centre will generally be quite remote, and too acutely determined for accuracy; hence proceed as follows. The contact, g''', of the two arcs may be found by an application of the problem: To draw a line through a given point, which shall pass through the intersection of two given lines. Thus, construct any triangle, as O″P1, of which O″ shall be one vertex; the two others being on B‴P and 1*i*, the perpendicular at the middle of O″P. Then draw 2, 3, parallel to OP; 3, 4, parallel to P1; and 4, 2, parallel to O″1, will meet 3, 2, in a point, 2, of the required radius, O″2, which limits the arc B‴g''' at g'''. Having thus found g''', we can, when,

as in practice, making the drawings on a very large scale, find the arc $B'''g'''$ by points as in (30).

4°. *Test for Interference.* The next step is to ascertain whether the surface, generated by the gate-top in opening, interferes with the top of the recess. The former surface is, for the left-hand gate, for example, a portion of the annular torus generated by the revolution of the circle of radius $O''H'$ around $H - H'H''$ as an axis. In such a torus, the gorge or interior opening reduces to nothing, and the portion used in the problem is generated by the quadrant $H'G''$. The proposed test is made by taking *horizontal* planes, as $Q'a'$, and finding whether their intersections with the two surfaces intersect within or without the jamb AH. Thus the plane $Q'a'$ cuts the gate torus in the horizontal circle of radius HQ, and the top of the recess in the curve abQ, found by projecting a', b', etc., upon the horizontal projections, lc, LA, etc., of the elements which contain them. When the circles as Qd, described by points, as QQ′ of the gate, everywhere intersect the curves, like Qba, cut from the recess roof by the respective planes of these circles, outside of AH, as at d, no interference exists. But when, as at n, the curve Lfe and circle Lne intersect within the jambs, there is an interference. In the latter case, one or more of four means may be used to remedy the difficulty: —

1*st.* To increase the radius $K'O'$, estimated from K'.

2*d.* To raise the point K', without increasing the radius $K'O'$.

3*d.* If the interference is very slight, a portion of the recess roof may be hewn out to coincide with the torus generated by $H'G''$, without disfiguring the surface by abrupt or too obvious changes of form.

4*th.* The radius of the gate-top may be slightly diminished.

II. *The Directing Instruments.* The construction of these will be best illustrated by showing patterns of all the developable surfaces of the most irregular one of the voussoirs, viz., that between the radial joints $q'o'$ and $r'u'$.

Besides the straight edge, and square, and the rectangular patterns of the top, and of the vertical side, at $u'v'$, of this stone, Nos. (1–4), there are, No. 5, a pattern of the back end, $q'o'v'u'r'$; No. 6, that of the front end, shown at $B'm'o'v'u't'$; No. 7, that of the radial joint at $r'u'$, which is shown by re-

volving it into the horizontal plane C_1B'', after supposing the back, C''D, of the wall to coincide with the vertical plane. Then $E'r_1 = rr''$; c_1g_1 = the perpendicular distance of g from C''D; $T_1t_1 = CC''$, etc. No. 8 shows, in like manner, the joint in the plane $OO''o$; and No. $9 = B_1g_1t_1$, Fig. 46, the true size of the surface $Ig - B'g't'$, forming a part of the jamb IB.

Bevels may also be provided, set to as many of the diedral angles, as $m'o'v'$, $m'q'r'$, etc., as may seem best. Nos. 10, 11, etc.

III. *Application.* Form the plane rectangular top, to the dimensions, $o'v'$ and CC'' of pattern No. 3; then the three vertical plane surfaces, viz., the two ends, and the side on $u'v'$, square with each other and with the top, and shaped by their patterns, 4, 5, and 6. Work the radial beds square with the back, just completed by No. 5, checking their positions by the right section bevels, 10, 11, etc.; and scoring their edges by patterns 7 and 8. The portion, $E'r_1s_1l_1$, determines the cylindrical surfaces on $r'q'$ and $p's'$, and the annular plane portion, $p'q'r's'$.

Every edge of the warped portion of the stone being now determined, this surface can be wrought by the straight edge, No. 1, held in the direction of elements of the surface, and these will be found by transferring their extremities as k' and h', M' and B', from the drawings to the stone.

THE OBLIQUE ARCH.

60. This, the most extended of all the problems in Stone Cutting, is usually made the subject of a separate treatise; for which its many and marked varieties, as well as its complexity, make it sufficient. Yet, by the full and careful exhibition of all the *essential* features of its *usual* form, the student can be prepared both to design and superintend the construction of an oblique arch as commonly built, and to read the works in which the subject is treated more elaborately.

PRELIMINARY TOPICS.

Elementary Mechanics of the Arch.

61. Let ABC be an ordinary semi-cylindrical arch, of which we will first consider only one half, as ABT*a*, Fig. 4. Let G

represent the centre of gravity of this half, and GH its weight, acting vertically downward. The half ABT, as a whole, and

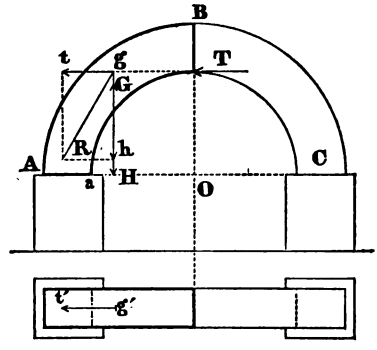


Fig. 4.

thus actuated by its unresisted weight, would fall, by turning about *a* as a centre. As it would prevent the use of the arch to oppose GH by props underneath, it is counteracted by a horizontal force acting, as at T, at any point of BT, and this horizontal force consists in the reaction of the other half, BTC, of the arch and its immovable backing. This understood, as a force may be considered as acting at any point on its own proper line of direction, the pressure at T, and the weight concentrated at G, may be considered as both acting at *g;* the former at *gt*, the latter at *gh*, whence *g*R would be their resultant. Now it is necessary for the stability of the half arch that *g*R should intersect the base of the arch between A and *a*, to prevent rotation about one or the other of those points; and what is thus evident for the half arch as a whole, is true of the separate stones of which it is composed. That is, by considering the successive segments from BT to the successive radial joints of the arch, in the same manner as just explained for the whole, we should find a series of forces like *g*R, one for each segment, and whose intersections would form a polygon, called the line of resistance, which should lie wholly within the arch in order to secure its stability.

62. Passing to the *oblique arch* (25), Fig. 5, it is evident from the foregoing explanations that the portions, ABC and ADEC, of the left half, are only more or less imperfectly supported by the opposite half. Hence, in discussing the oblique arch relative to its stability, it is usual to consider it as divided

by planes parallel to a face, as GH, into an indefinite number of laminæ; each of which will be a right arch of a span equal to *ab*, the oblique span of the given oblique arch.

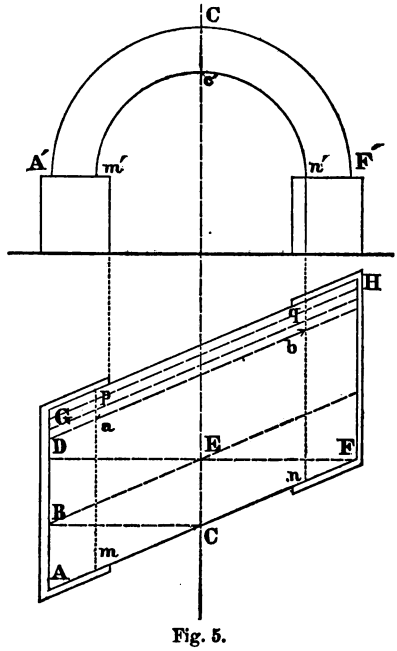


Fig. 5.

That is, *the "thrust," "lines of pressure," or "lines of resistance" in an oblique arch, are assumed to act in planes parallel to its faces.*

The resulting standard or essentially perfect design for an Oblique Arch.

63. The conclusion of the last topic affects the form and disposition of the joints of an oblique arch, and thence their graphical construction, in the following manner: —

When two surfaces are pressed together, the pressure at each point should act in the direction of the normal to the surfaces at that point. Else it can be resolved into two components: one, normal to the surfaces; and one parallel to them, which will tend to produce slipping.

Thus, if the surfaces of two bodies meet in the plane whose trace upon the paper is AB, Fig. 6, and are acted upon by a force producing a pressure at any point, *p*, which may be rep-

resented by $OP = pP_1$, this pressure may be decomposed at any point as p of its line of direction into the normal com-

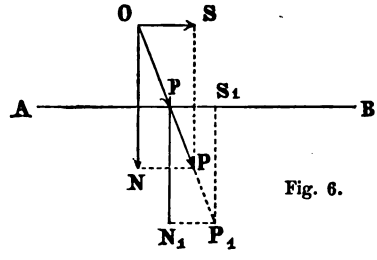


Fig. 6.

ponent $pN_1 = ON$; and the parallel component $pS_1 = OS$; which last will tend to produce slipping in the direction pS_1.

64. In applying this principle to the design of the joints seen on the intrados (19) of an oblique arch, the "*transverse*," "*heading*," or "*broken*" *joints* are made in planes parallel to the faces, and thus represent on the arch itself the direction of its thrust. The "*longitudinal*," "*coursing*," or "*continuous*" *joints* are then made so that each shall intersect at right angles all the transverse joints which it meets.

65. On account of the rectangular intersections of the joints, arches thus designed are often described as forming the *orthogonal system*. On account of the equilibrium of the pressures thus acting in them, they are also often called *equilibrated arches*.

The coursing joint is also often called the *trajectory*.

Problem XII.

The partial, and trial construction of the orthogonal or equilibrated arch.

66. Construction of a coursing joint.—1*st*. *In vertical projection.* Let the plane V, Pl. VI., Fig. 48, be considered as parallel to the faces ABC and DEF of the arch. These semicircles, with the equal ones having any convenient number of equidistant points o, o_1, o_2, etc., on DC, as centres, represent the vertical projections of sections of the arch parallel to its transverse joints (64). Now, *first*, a line is *normal* to a curve at a point, when it is perpendicular to the *tangent* at that point; and, *second*, as we learn from descriptive geometry, if

one side only of a right angle be parallel to a plane, the projection of the angle on that plane will be a right angle. Hence in the figure, as the *tangents* to the circles ABC, etc., are parallel to V, the vertical projection of the normal curve to all these circles will be perpendicular to the projections of their tangents at its intersections with these circles. Thus, if a_0ac_1 represent an arc of the curve, it will be *perpendicular* at a to the tangent aT, and therefore *tangent* at a to the radius ao. Hence we have the following construction: —

Let a be one of the points of division of one face, ABC, of the arch, through which a coursing joint is to pass. Draw the radius ao; from a_1, its intersection with circle o_1, the radius a_1o_1; from a_2, intersection of a_1o_1 with circle o_2, draw the radius a_2o_2; from a_3, similarly found, the radius a_3o_3, etc.; and the line $aa_1a_2 \ldots . a_n$, thus found, will nearly coincide with the required curve; only that all the points except a are evidently a little *too low*. It is an advantage, however, that the errors are wholly on one side of the truth.

Now draw ao_1, meeting circle o_1 at c_1; thence c_1o_2, likewise giving c_2; thence c_2o_3, etc., and again, $ac_1c_2 \ldots . c_n$ will be nearly the required curve, but all its points besides a will be a little *too high*. How now shall we approximate more closely to the correct curve?

67. At this point a general observation upon the relative exactness of constructions made by plotting to scale the results given by analysis from numerical data, or by geometrical construction from graphical data, may be useful. The finest lines, as perfect both in *length* and *direction* as can be laid down on *a large scale*, and with the nicest care, on firm and smooth paper, will give results equal, in nearly or quite every case, for all practical purposes, to those of analysis.

The perfect length and location of lines are analogous to absence of errors of calculation; and fineness and large scale of construction are analogous to the carrying out of numerical computations to a large number of decimal places.

Besides, in all cases where drawings are necessary as guides in the execution of a work, the results of calculation are exposed to instrumental errors in plotting them to scale.

68. Returning now to the means of approximating to the true curve, any one or more of the following methods may be employed.

1°. *Increase to any desired extent the number of circles* between the faces, ABC and DEF, of the arch; since this will evidently cause the curves aa_5 and ac_5, which are on opposite sides of the true curve, to approach each other; that is, will cause either of them to approach the true curve.

2°. Bisect the spaces a_1c_1, a_2c_2, etc., and the curve through the points thus formed will sensibly coincide with the true curve; especially in a figure drawn on a large scale, and with numerous auxiliary circles.

3°. Bisect the distances oo_1; o_2o_2, etc., between the centres of the successive circles ABC, etc., at the points 1, 2, 3, etc., and draw $a1$, which will evidently divide a_1c_1 at some point as n_1, not shown. Then draw n_1, 2; meeting a_2c_2, at a point n_2, etc., and $a\ n_1\ n_2$, etc., will very nearly coincide with the true curve.

4°. Suppose the point n_1, just described, to be a point of the true curve. Then, observing that the curvature of the required curve rapidly increases as it ascends, tangents to it at a and n_1 would meet nearer to a than to n. Hence assuming t as one point of a required curve, draw to_5, and take r on the part ts and a little nearer t than to s; and draw ro_4, and on the portion t_1s_1 of this line take r_1, nearer the middle of t_1s_1 than r was to the middle of ts; thence draw r_1o_3, and on the portion t_2s_2 of this line take r_2 nearer the middle of t_2s_2 than r_1 was to the middle of t_1s_1; and so on; and tt_1t_2, etc., will be very exactly the true curve, tangent at t, t_1, t_2, etc., to to_5, ro_4, r_1o_3, etc.

EXAMPLE.—Work out each of the above four methods with the circles ABC, etc., drawn to a radius of at least three inches.

69. Horizontal projection of a coursing joint. — Assuming merely for illustration that ac_5 is the true vertical projection of such a joint, simply project its points upon the horizontal projections of corresponding circles, as a at a'; c_1 at c_1'; c_2 at c_2', etc.; where the *accents are given to the horizontal projection*, as is sometimes done, *when the vertical projection is first made*, or of most service.

70. Development of a coursing joint. — Let the cylinder A′C′D′F′ be first made tangent to the horizontal plane along the element, CF — C′F′, and then rolled out upon that plane. All its elements will then be parallel to C′F′ in development, and at distances from it equal to the true lengths

of the arcs of *right section* between them and C′F′. Hence make $m''n'$ equal to the true length of the elliptic arc $m'n'$—mn of right section, as it would appear *projected upon a plane* perpendicular to the axis OO_5 — oo_5. (This construction is not shown, for want of room). Likewise $v'a_0''$ equals the true length of the elliptic arc (arc of right section, whatever it may be, in any case) $v'a_0''$—va_0; and $q'a''$ equals the true length of $a'q'$—aq; etc. for any sufficient number of points: Then $p'm''a_0''$; $C'a''$; $q'c_1''$, etc., are the developments, all alike, of the equal semicircles, $a_0'p'$ — a_0p; OC' — oC; etc.; and $a_0''a''c_1''c_2''$, etc., is the development of the coursing joint $a_0'c_5'$ — a_0c_5; and it crosses the developed semicircles at right angles.

71. *Identity of form of the coursing joints.* — Through any point, as a, Fig. 48, draw an element meeting the other circles at points as b, homologous with a. It is perfectly evident from (70) that, if we construct the trajectory passing through b, as in (66), it will be of the same form as that through a. That is, all the trajectories have parallel tangents at points on the same element, and hence are all alike.

Likewise, the horizontal projections, and the developments of like portions of the trajectories are identical in form. Hence having any projection or development of any trajectory, the portion of any other one included between the same elements of the cylinder would be drawn simply by a card-board pattern of this initial one.

72. *Convergence of the coursing joints.* — It will be immediately evident on constructing any other coursing joint, as the one through $m'm$ for example, that these joints converge, rapidly at first; as they approach CF — C′F′ in the direction from C′ to F′, and DA — D′A′ in the direction from D′ to A′ when produced indefinitely in both of these directions.

The result of this convergence is, that no two stones in the same course are alike; though stones in like positions in different courses are so. This greatly increases the number of necessary patterns, difficulty of execution, and the expense of constructing an equilibrated arch.

73. *Generation of the joint surfaces.* — We know that 1°, the *normal* to a surface at any point is perpendicular to the tangent plane at that point; 2°, that when a line is perpendicular to a plane, the projections of the line are perpendicular to the

traces of the plane. Now aT is the vertical trace of the tangent plane to the intrados along the element ab, at aa'; its horizontal trace would be a line through T, parallel to C'F'. Hence the normal to the intrados at aa', for example, is oa — $a'a''$. If then an arm de, normal, at c_2', be fixed to a rod O_1O which coincides with the axis OO_5, and if the rod and arm move together, the rod moving so as to coincide with the axis, and the arm, so that the point c_2' of the arm, shall continue on the semicircle O_2u, the point d, considered as one point of the extrados, will generate the *transverse joint of the extrados*, corresponding to the semicircle O_2u of the intrados; and the portion $c_2'd$ will generate one of the transverse joint or heading surfaces; which is thus everywhere *normal to the intrados*.

Likewise, the rod O_1O continuing to coincide with the axis as it moves, with the arm, let the pair move so that the point c_2' of the arm shall trace the coursing joint $a'c_5'$, the point d will then trace the extradosal trajectory corresponding to $a'c_5'$ of the intrados, and the line $c_1'd$ will generate the normal joint surface having $a'c_5'$ for its directrix.

74. *Nature of the joint surfaces.* — These are evidently both warped, but of a variable twist; the former, the transverse one, is so by reason of the *variable relative velocity* of the *axial* and the *rotary* motions of the frame O_1ed, which is immediately evident on constructing any three of its positions; the latter, the coursing joint surface, is variably warped by reason of the variable curvature of the coursing joints.

75. *Summary.* — Without detailed drawings, or further investigation, five grave practical objections to the equilibrated arch are already apparent.

1°. The *inequality* of the stones in each course, resulting in the evils already noted (72).

2°. The *variable twist* of the normal joint surfaces, which still further complicates their execution.

3°. The non-parallelism of opposite faces of the same stone, whereby the actions and reactions upon those faces are not in the same line, and therefore in the aggregate yield resultant couples, tending to produce rotation of the arch about a vertical axis.

4°. The *convergence of the coursing joints*, while like points of each are not, as we have seen, in the same section parallel to the face, also makes the stones in the face of unequal width,

unless they are arbitrarily made equal by breaking joints with those behind them.

5°. The same convergence, finally, renders it impossible to build equilibrated arches of common brick, as is often desirable.

For these reasons, we shall here dismiss the equilibrated arch, only referring the reader, who wishes to become familiar with its details, to the works of *Bashforth*, *Graeff*, and ADHEMAR; and shall seek an approximation, which, as nearly as possible, retains the merits, but avoids the defects of the equilibrated arch.

We will begin by seeking a simpler form of warped surface, normal to the cylindrical intrados, for the heads and beds of the voussoirs.

The Helix.

76. Let $o - 0'$, 12', Pl. VII., Fig. 50, be a fixed axis, and let 0,0' be the initial position of a generating point which has these two *simultaneous* motions; one of *revolution* around that axis, and one of *translation*, parallel to it. By the *first motion alone*, the point, 00', would generate the circle 0,3,6,9,12; by the *second alone*, the straight line, $0 - 0'12'$; but, by both motions combined, it would generate the curve called a *helix*. The height 0'12' corresponding to a full revolution, 0,3,9,12, is called the *pitch* of the helix.

The usual case is, that each of these motions is *uniform*, giving rise to the *common* helix; or simply the helix, as commonly understood.

77. *Three elementary properties* immediately follow from the *definition* just given.

1°. The helix will lie on the convex surface of a *cylinder* of *revolution* whose axis, $o - 0'12'$, is that of the helix; and whose radius is the perpendicular distance $0o$ of the generatrix 00' from the axis.

2°. Since the two component motions of the generatrix, one around the cylinder at right angles to its elements, and the other, parallel to them, are each uniform, the path of this point, *i. e.* the helix, must *cross all the elements at a constant angle.*

3°. It now follows that when the convex surface of the cyl inder is developed, the *development of the helix will be straight*, for the elements of the cylinder will be parallel in development

and a line which crosses parallels at a constant angle must be straight.

It thus appears that the joints, CD_1, KQ_1, etc., B_1IV, C5, etc., Fig. 49, primarily chosen as convenient and sufficient substitutes for the theoretic ones of the orthogonal system, are *helices*.

78. The construction of the helix also follows immediately from its definition (76), as is shown in Pl. VII., Fig. 50.

For let the distance 0′12′ on the axis, be the height attained by the generatrix while also making one revolution around the axis from 0,0′ to 12,12′. This revolution is evidently indicated in horizontal projection by the circle of radius $o0$. Then, as both motions are uniform, divide this circle, and the corresponding height (pitch) 0′12′, into the same number of equal parts, and draw horizontal lines through the points of division on the axis, and number them, as shown, to correspond with the numbers of their horizontal projections, $o0$, $o1$, etc. Then project up 1, 2, 3, etc., from the plan upon the lines similarly numbered in elevation, at 1′, 2′, 3′, etc., and the curve 0′, 1′, 2′, 3′, etc., will be the vertical projection of the helix.

Theorem II.

The projection of the helix on a plane parallel to its axis is a sinusoid.

A sinusoid is a curve in which if the *abscissas*, $0'b'$, $0'g'$, etc., are equal to, or proportional to the arcs of a given circle, the corresponding ordinates $1'b'$, $2'g'$, etc., are equal to the sines of those arcs. Now $1'b' =$ the sine of the arc 01 in the plan; $2'g' =$ that of the arc 02 in the plan, etc.; and the spaces $0'b'$, $0'g'$, etc., are proportional to those arcs. Hence, the curve 0′, 1′, 2′, 3′, ... 12′ is a *sinusoid*.

The Helicoid.

79. Let now all the points of the horizontal line $o0 - 0'$, Pl. VII., Fig. 50, have the same two simultaneous motions that have just been given to its outer point, 00′. Every point of the line $o0 - 0'$ thus describes a helix, having the same pitch, 0′12′, and whose horizontal projection would be a circle centred at o. All the consecutive helices, from the axis 0′12′, to 0′, 1′, 2′,

3′ . . . . . 12′, and so on outward without limit, thus described, will constitute the surface called a *right helicoid.* The portion within the cylinder of radius *o*0, is shown by the shaded area of Fig. 50.

80. The surface just defined is called a *helicoid*, because its *generatrix*, the line, *o*0 — 0′, moves upon a given *helix*, as 0′, 1′, 2′, 3′ . . . . . 12′, as a *directrix;* and a *right* helicoid because this generatrix is perpendicular to the axis *o* — 0′12′. The surface is the same as that of the screw surfaces of a square threaded screw, or of the plastering on the underside of circular stairs.

The *right helicoid* is evidently *normal* to the convex surface of the *right cylinder having the same axis*, since all its elements are perpendicular to the axis of that cylinder.

Thus we see that the *joint surfaces* of the voussoirs (18) normal to the cylindrical intrados of the oblique arch are *right helicoids.*

Problem XIII.

A segmental oblique arch, on the helicoidal system.

I. 81. *The Projections. Primary outlines.* — Let ABCD, Pl. VII., Fig. 49, be the plan of the intrados of a segmental (27) arch having a *square span*, AG, of 13 ft.; an *angle of skew*, CAB, of 54°; and a perpendicular length, yy_1, of 14 feet, between the vertical planes, AB and CD, of the faces.

Let EF — O″ be the projections of the axis of the arch, taken perpendicular to the plane **V**, and let the arc A′E′B′ of 120°, centred at O″, indicate the extent of the segment of the cylinder of revolution, which forms the intrados.

Then AB_1D_1C is evidently (see Pl. III., Fig. 28, etc., the development of the intrados; where $CC_1 = 15'.708 =$ the length of the right section Cc_3 — A′E′B′; and the curves AB_1 and CD_1 are the developments of the face lines, AB — A′E′B′, and CD — A′E′B′; as will be presently explained more in detail.

This being understood, and the curve CD_1 representing, at each point, the direction of the thrust of the arch at that point (62) the *straight* line CD_1 symmetrical with the curve, so nearly coincides with it as to sufficiently replace it as *a proper direction for the developed transverse joints of the intrados;*

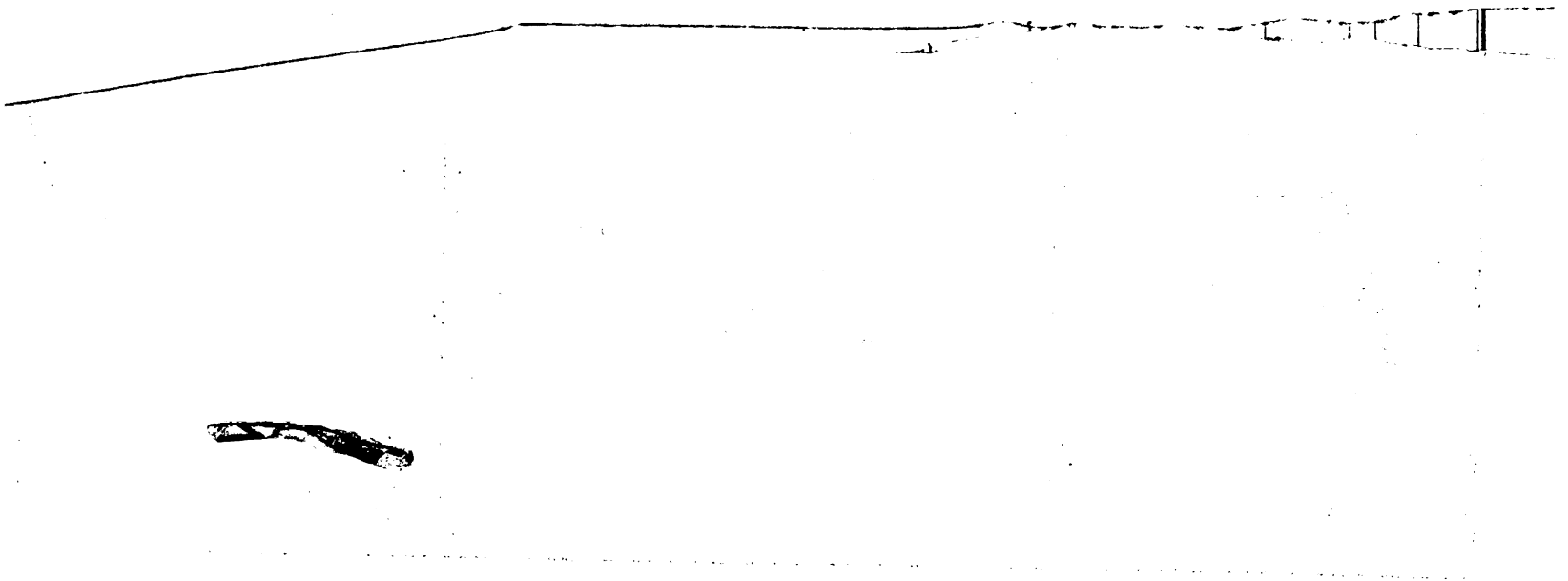
but, being straight, is evidently the development of a helical arc (77, 3°).

This settled, next divide the straight line CD_1 into the odd number of equal parts, chosen as the number of voussoirs, in the width of the arch, in this case nine. Then, as the transverse and coursing joints should be nearly or quite at right angles to each other, let the developed coursing joints, B_1IV, etc., be perpendicular to CD_1, or as nearly so as the length D_1B_1, or width CC_r, if unalterable, will allow. Such coursing joints, being evidently both parallel and equidistant, the arch may be *built of brick* if desired, (75, 5°) and, if of stone, all the voussoirs in *all* the courses will be of equal width (75, 1°, 3°, 4°.)

Returning to Fig. 50, and comparing Figs. 49 and 50, the axis of the arch replaces the axis $o - 0', 12'$, and is perpendicular to V. Hence, A′E′B′ is the vertical projection of the right cylindrical surface of the intrados, and of all the helices traced upon it; and the *horizontal* projections of these helices will be similar, in form and construction, to the *vertical* projection of the helix in Fig. 50.

82. *Horizontal projections of transverse helical joints on the intrados.* — KQ_1, parallel to CD_1, being the development of one of these joints (77, 3°), and Kk_4 that of the right section at K, shows that Q_1k_4 is the partial pitch due to the arc A′E′B′ of the circular projection of the same helical joint. By counter development, Q_1k_4 returns to Qk_3. Hence, if we divide A′E′B′ and Qk_3, each into the same number of equal parts, numbered similarly from B′ and Q as zero points, the points of the horizontal projection, KPQ, of the helix considered, will be at the intersection of the projecting lines from 1′, 2′, 3′, etc., on A′E′B′, with parallels to Kk_3, through the corresponding points on Qk_3, as shown in the figure. Thus, M_2 and B_3 are projected from 2′ and 3′ upon the second and third parallels from Q.

83. *Horizontal projections of intradosal helical coursing joints.* — In like manner, R_1S_1T, one of the parallels drawn through points of equal division on CD_1 is the development of so much of coursing joint as lies on the given segment of the cylinder taken to form the arch. Tr_5 is, therefore, its proportion of the pitch (76) of a coursing joint, and A′E′B′ is its vertical projection. Then, as before, divide Tr_5 and A′E′B′ into the same number of equal parts, number similarly from B′ and r_5 as zero points,



and the intersections of projecting lines from 1′, 2′, 3′, etc., with the parallels to Rr_5 through the like numbers on Tr_5, will be points R . . . R_2, R_3, etc. of the coursing helix RST.

84. *Construction of horizontal projections of helices from their developments.* — The above constructions, having been inserted to render them more intelligible by their analogy with Fig. 50, it will now be shown that the sinusoid (Theor. II.) projection of the helix can be found from its circular projection and development. For the same parallels that divide $Q_1k_4 = Qk_5$, the pitch of KPQ — A′E′B′ into equal parts, divide the development, KQ_1, of the same helix into the same number of equal parts ; and the like is true of R_1T. Hence, in cases like the present, where the developments of the helices of the intrados are given, their horizontal projections, as QPK, are most naturally found as the intersections of the projecting lines from 1′, 2′, 3′, etc., on A′E′B′, with perpendiculars to the axis EF, from the developments of the same points, O_4, M_3, B_4, etc., thus giving respectively O_3, M_2, B_3, etc., on QPK.

All the other intradosal helices are found in a precisely similar manner, as is now sufficiently evident by inspection.

85. *The development of the face lines of the intrados.* — *1st Method.* Parallels to AC through the points of division 1, 2, 3, etc., on AB_1, are the developments of the *elements* whose vertical projections are the vertical projections 1′, 2′, 3′, etc., of the same points of division. The horizontal projections of the extremities of these elements are found by projecting 1′, 2′, ... 6′, 7′, etc., upon both face lines, AB and CD, as at 6 and 7 on AB, and 7 on CD. Thence, in being developed, they pass in planes 6 — z_1 ; 7 — z_2, etc., perpendicular to the axis EF, to z_1, z_2, etc., of AB_1, and from 7, etc., on CD, to VII, etc., on CD_1, upon the developed elements, 7 VII, etc.

86. *Construction of the developed face-line as a sinusoid.* (Theor. II.) If the equal divisions of A′E′B′ be projected upon AB, as at 6, 7, etc., these points will divide AB in a certain manner. If the latter points be thence projected upon G_1B_1, projection of AB upon G_1D_1, the lines AB and G_1B_1 will be divided similarly ; hence, if $B_1m_5G_1$ be an arc similar to A′E′B′, and on the chord B_1G_1, it will be divided equally by the parallels to GG_1, through the points B . . . . 6, 7, etc., on AB. Hence, we may regard the arc $B_1m_5G_1$, and the straight line B_1A, as respectively the circular projection and the develop-

ment of a helix, points of whose sinusoid projection, B_1A, are found as the other like curves, KPQ, have been, from like data; viz., at the intersections of parallels to GG_1, from m_1 . . . m_6, m_7, etc. with parallels to G_1B_1 through like points, 1, 2, 3, ... 6, 7, etc., on the straight line B_1A.

87. *Useful limits of the arc of the intrados.* — These are evident from the development while considering the helical joints. Were the arch extended to a_3a_4 — A_0A_4, so as to become full centred (27), CD_1 would be extended each way, as shown at D_1D_2, where $D_1d_2 = A'A_0$, and $D_2d_2 = a_3v$; and the angle at D_2 is 90°. Hence, we see that parallel coursing joints, quite nearly perpendicular to the face line between C and D_1, as is desirable, — become more and more oblique to it as we approach D_2 on D_1D_2. Moreover, as can readily be imagined with a given case in view, the *greater* the obliquity, that is the *more acute* the angle CAB, the *less* should be the segment taken to form the intrados.

88. *The extrados.* — While the outer surface of the actual arch would be left rough, yet it is convenient to represent an ideal extrados, or extrados of construction, to aid in forming the voussoirs.

Such an ideal extrados is a cylindrical surface having the same axis as the intrados, and terminated by the same vertical planes of the faces. Its projections are, therefore, $A_1B_2C_1D_2$, and the arc $A_2'F'B_2'$ concentric with $A'E'B'$. Then A_2C_2 and B_2D_2 are the *outer* springing lines.

The intrados and extrados being thus concentric cylinders, the point, as qB_2', in which the generatrix, Qq — $B'B_2'$, perpendicular to the axis EF — O'' (79), of any of the helicoids intersects the extrados, will generate helices upon the extrados, whose developments and horizontal projections will be found in the same manner as has now been explained for the intrados.

89. *Development of the extrados.* — As the helicoidal surfaces of the voussoirs are *right* helicoids (79), their elements are perpendicular to the axis EF — O''; hence Kk and Qq, perpendicular to EF, are such elements, and the *outer* helix (helix on the extrados) corresponding to the *inner* helix (helix on the intrados) KPQ will extend from k to q. Hence, making $n_9a_2 = A_2'FB_2' = 20'.94$ (77, 3°) a_2c_1, parallel to B_2D_2 is the developed position of A_2c; and, carrying k across to this

line at k_1, gives qk_1 as the development of the helix from q to k.

The other developed *transverse* helices, as ba_1 and dc_1, are parallel to qk_1, and are similarly divided, to give the developed extradosal *coursing* helices bd_4, and the parallels to it, corresponding to B_1IV, and its parallels on the intrados.

90. *Contrast between the intrados and extrados.* — Here we meet with two points of difference. *First* — As the pitch, Qk_3, is the *same* for both of the helices, KQ_1 and qk_1, while the latter is longer, the angles o_1qo and o_1oq are *greater* than the corresponding angles $O_4Q_1O_1$ and $Q_1O_1O_4$ of the intrados. Hence qo_1o and its equals at all the intersections of helices on the extrados, are *less* than $Q_1O_4O_1$, and all like angles on the intrados.

Second — As the *outer helix*, corresponding to an inner one from C to D, connects c with d, while the *outer face line* is C_2D_2, corresponding to the inner one CD, the developed extradosal face line, D_2c_2, and extreme helix, dc_1, do not terminate at the same points, as they do on the intrados where the like lines are $C_{VII}D_1$ and $C7D_2$.

91. *An interesting consequence* of the *first* of the preceding differences is, that to equalize the angles qo_1o, and $Q_1O_4O_1$, so that the helicoidal surfaces should be normal to each other somewhere between the intrados and the extrados, the initial coursing joint, B_1IV, should be drawn to a point, IV, on the side towards D_1, next to x, the foot of the perpendicular from B_1 to CD_1, whenever x does not coincide with one of the points of division of CD_1.

92. *The construction of the developed face lines.* — This may be either as at $1'''$, etc., by projection from $1''$, etc., to 1, etc., on CD, and transference thence, by perpendiculars, to BD, to elements through n_2', etc.; or as at B_2a_2 by the sinusoid projection of a helix, from $B_2n_5n_9$, and ba_1, the latter substituted, without affecting the result, for a straight line, B_2a_2, as its circular projection and development. Either method is obvious on inspection, in connection with the like constructions of the developed face lines of the intrados.

The construction of the horizontal projections $k\mathrm{P}q$, etc., and $r\mathrm{S}t$, etc., of extradosal helices, is now obvious by inspection. Thus m_2 is the intersection of the projecting line $m'm_2$ with $m_3'm_2$ perpendicular to EF.

93. *Convenient checks* upon the accuracy of the horizontal projections and developments of the helices and face lines occur as follows: *First* — The necessary intersection of corresponding inner and outer helices on EF, in horizontal projection, as at L, S, P, and N, because the elements of the right helicoids containing such helices are vertical at these points. *Second* — The developed positions as L_1 and l_1, or N_1 and n_1 of such points are in the same line, L_1Ll_1, or N_1Nn_1, perpendicular to EF, and on E_1F_1 and e_1f_1, the developments respectively of EF — E′ and EF — F′. *Third* — As helices from C to D and from *c* to *d* necessarily cross EF at F, the middle point of CD (as in Fig. 50 the helix crosses $0'_1 12'$ at its middle point, 6′), their developments, CD_1 and dc_1, cross; the first, $CVIID_1$ and E_1F_1 at F_1, the middle point of $CVIID_1$; and the other, dc_1 and e_1f_1 at f_1, the middle point of dc_1.

94. *Completion of the abutments.* — These being alike, the construction is shown only on one. The back of the abutment is properly stepped by vertical planes, parallel and perpendicular to that of the face (which represents the direction of the thrust (62) of the arch), as shown at oZ_1, and mZ_1. Then, having regard to symmetry, and to the protection of the corners of the abutment, make $mZ = oZ_1$, and draw BZ; and, for the opposite end of the abutment, make the angle $D_2Dz = ZBB_2$, and $Dz = BZ$. These last details being, however, unessential, may be varied at pleasure.

95. *The face joints.* — These are the intersections of the planes of the faces with the coursing helicoids.

Now, referring to Fig. 50, as all the elements intersect the axis, and are perpendicular to it, any plane containing the axis or a perpendicular to it at any point of a helix, will contain an element of the helicoid.

Comparing with Fig. 49, the plane, AB, of the face for example, does not contain S, P, or L; hence, it does *not* contain elements of the helicoids SR*r*, K*k*P, or M*m*LI, which it therefore intersects in curves. It intersects the coursing helicoid SR*r* in the face joint $R'''r'''$.

96. *The direct construction* of the face joints will then be, to project the points as R‴ (derived from R⁗, extremity of a coursing joint on the developed inner face line) at R′; and r''', likewise derived from r_1''' on the developed outer face line, at r'; when R′r', though straight, would very nearly be the

proper face joint. And so we might operate to find all the face joints.

97. *Representation by tangents.* — The face joints are so nearly straight, especially in a segmental arch, that their tangents, at their inner extremities, may often be sufficient to represent them. Now we know from descriptive geometry: first, that the tangent line at a given point, t, of the intersection of any plane, P, with a surface, S, is the intersection of the plane P with the tangent plane to the surface S, at the point t; second, that the tangent plane to a helicoid, at a given point, is determined by the *element* through that point and the tangent to the helix through that point.

98. *The following construction* depends on the principles just given. Let the tangent at R'''R' be constructed. *First* — AB is the plane of the curve, and its trace on the plane, EE_1, of right section is E — O''F'. *Second* — The tangent plane at R'''R', to the helicoid, RrS, is determined by the element R'''L_2 — R'O'', and the tangent, at R'''R', to the helix, RS — B'R'E. Now the tangent to a helix at a given point lies in the plane which is tangent to the cylinder containing the helix, along the element containing the given point, and it makes the same angle with that element that the helix does. But as the latter angle is constant for all the elements, the development of the helix, upon the tangent plane must coincide with its tangent in that plane.

Hence the tangent at R' is the vertical projection, and R''''R'', coinciding with S_1R_1, is the development of the tangent *line* at R'''R'; and R''W'' is the projection, upon EE_1, of the portion R''''R'' of this tangent. Hence make R'W' = R''W'', and W' will be one point of the vertical trace, on the plane EE_1, of the tangent plane to the helicoid RrS, at R'''R'. But the *element*, R'''L_2 — R'O'', is parallel to the vertical plane EE_1, hence W'U', parallel to R'O'' is the vertical trace of the auxiliary tangent plane. This meets E — O''F', the like trace of the plane of the curve, at U', which is therefore one point of the *required tangent line*. The given point of contact R'''R' is another; hence ER''' — U'R' is the tangent, at R'''R', to the face joint through that point.

99. *Focus of like Tangents to Face Joints.* — Draw R'X, perpendicular to the horizontal O''X; and O''U_1 perpendicular to U'W' and hence to O''R'. The angles at U' and R' in the

triangles $O''U_1U'$ and $R'XO''$ thus formed are therefore **equal**, and those at U_1 and X are equal, being right angles. **Hence** these triangles are similar; whence we have: —

$$O''U' \cos. U'O''U_1 = O''U_1 \quad (1)$$

Also, $$O''U_1 = R'W' \quad (2)$$

Remembering that $R'W' = R''W''$; and calling the **pitch**, $= 3Tr_5$, of the entire helix, 3RST, $= h$, we have

$$R'W' = \frac{2\pi O''R'}{h}. R'''W''. \quad (3)$$

From the triangle $R'''EE_2$; $R'''E_2 = EE_2$ tang. $R'''EE_2$ (4)

But, $$EE_2 = O''X. \quad (5)$$

and $O''X = O''R' \cos. R'O''X$; or, $O''X = O''R' \cos. U'O''U_1$ (6)

Now cancelling all the terms which are common to the **two** columns of left hand and right hand members in these **six** equations, and multiplying together the remaining terms **in** like columns, we have:

$$O''U' = \frac{2\pi.\overline{O''R'}^2. \tan. R'''EE_2}{h} \quad (7)$$

Now in this equation, every term in the right hand member is constant, hence $O''U'$ *is constant.*

We thus find the interesting property that *the tangents to all the face-joints at their inner extremities, meet at a common point on the vertical line through the centre E, O'' of the face of the arch.*

The point U' is called the *focus*, and the distance $O''U'$, the eccentricity of the arch; or, more strictly, of the intrados; for it is evident that a similar construction exists for the *extrados*, or for all the points in *any* cylinder, concentric with the intrados.

100. *Condensed construction of Foci.* — U' being independent of the position of R' on A'E'B', take A_0a_3, the point in the horizontal plane of the axis, and draw through it a developed helix as a_3u meeting the plane of right section FF_1 (corresponding to EE_1) at u, then, projecting a_3 on FF_1 at o'', gives $o''u = O''U'$.

Likewise, for the *extrados*, take A_4a_4, corresponding to A_0a_3 in the intrados, and draw a_4v parallel to the developed extradosal helices, bn_1, etc., and project a_4A_4 on FF_1 at O_2, then O_2v

will be the eccentricity to lay off from O″, giving U″ as the focus for the extrados; so that, for example, $r'U''$ is the vertical projection of the tangent at $r'''r'$ to the face-joint $R'''r''' - R'r'$.

101. *Foci for any Cylinder concentric with the Intrados.* — If, in a given arch, we take such a new cylinder, only the terms $O''U'$ and $\overline{O''R'}^2$ will change; hence, as is readily obvious, calling R_0, R_0', etc., the successive values of the radius $O''R'$, and calling the consequent positions of the focus, U′, U″, etc. we have,

$$\frac{R_0^2}{O''U'} = \frac{(R_0')^2}{O''U''} = \text{etc.}$$

$$\text{Then putting } \frac{R_0^2}{O''U'} = c, \text{ we find } O''U'' = \frac{(R_0')^2}{c}$$

whence $O''U''$ is found simply as a third proportional. Thus having drawn $U'A_0$, and A_0T, perpendicular to it, gives $O''T$ (on $O''F'$ produced) $= c$; since $R_0^2 = (O''A_0)^2 = O''U'' \times O''T$. Then c being constant, draw for example TA_4, and A_4U'' perpendicular to it, and U″ will be the focus of $A_4F'B_2'$, which is chosen for illustration, to save additional lines. For,

$$(O''A_4)^2 = O''U'' \times O''T$$

$$(O''U'') = \frac{(O''A_4)^2}{O''T} = \frac{(R_0')^2}{c} \text{ as above.}$$

In this simple way we finally find the focus for any cylinder, and thence as many tangents as we please to each face joint in making an exact working drawing on a large scale. In the figure, then, the face joint $R'r'$ will be a curve, tangent to $U'R'$ at R′ and to $U''r'$ at r'.

102. *Curves of the Face-joints.* — To get an idea of these, see Fig. 50 again, where PQP', $P_1Q_1P_1'$ (P_1Q_1 not shown) and $P_2Q_2P_2'$ are three parallel planes cutting the right helicoid shown by the shaded area of the figure.

Auxiliary horizontal planes will intersect both the helicoid and any one of the given planes in straight lines, whose intersection, in each of the auxiliary planes, will be a point of the required curve.

Thus, taking the plane PQP', the horizontal plane $0'Q$ cuts from the helicoid the element $o0 - 0'$, and from the plane, the line PQ. These being parallel, meet only at infinity;

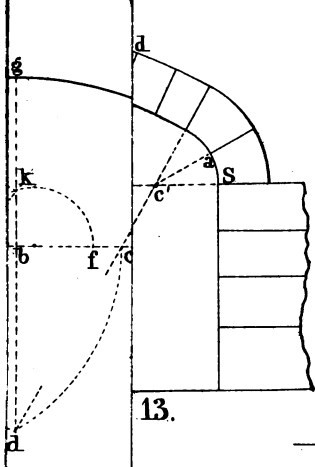
hence PQ is an asymptote to the curve. The plane $b'1'$ cuts from the helicoid the element $o1 - b'1'$, and from the plane, the perpendicular to V at c', which meets the element at $c'c$. Likewise, project d' at d, etc. Then at o' the intermediate plane $o'a'$ cuts from the helicoid the element $oa - o'a'$, and from the plane, the line $o0 - o'$ which meet at oo', showing that the horizontal projection of the curve passes through the centre of the circle 0–3–6–12. Next, the plane $6'q'$ cuts the helicoid and the given plane in parallels at $6'$ and q' respectively, which only meet at infinity, hence qr is another asymptote. The branch, *cdoq*, meets this asymptote at infinity towards q; while the new branch projected from $s', t', u' \ldots e', f'$ at $s,t,u \ldots e,f$ meets the same asymptote at infinity towards r, and the asymptote fk, in the plane, $12'f$, at infinity towards f.

Thus we see that the entire intersection consists of two branches; and that when the given plane, PQP', as here placed, cuts the helix *twice on the same side of the axis*, as at h' and v', the curve is shaped as at $s\ t\ e$, and does *not* pass through o; but that when it thus cuts the helix but *once*, as at the point a little below c', the curve does pass through o, in the horizontal projection.

103. *Applying this construction to the arch;* AB represents the plane PQP' of Fig. 50, and each coursing helicoid, in succession, will represent the helicoid of Fig. 50. Taking the helicoid, RrS, for illustration, the intersection of AB with successive elements parallel to Rr would, when projected upon the vertical projections, $B_2'B'$; $1''1'$, etc., of these elements, be the vertical projections of points of the indefinite face joint $R'r'$; and a parallel to EF, at s_2, the intersection of the plane AB with the plane Ss_1, would be an asymptote to this curve which would resemble *cdoq* in Fig. 50.

104. *Other projections.*—In finished drawings for exhibition or other purposes, elevations on planes parallel to the face, or to the axis, may be desired. These are easily made, as shown in Fig. 51, a fragment of the vertical section through the axis. A''E''B'' is the right section of the intrados, and the parallels to A''B'' through its points of equal division 1, 2, 3, etc., are the vertical projections of the elements at 1', 2', 3', etc., on the new plane.

The portion, RS, of the inner helix, RST, is then projected; R, at R_1'; R_2, at R_2'; S, at S_1', etc. Other helices could be



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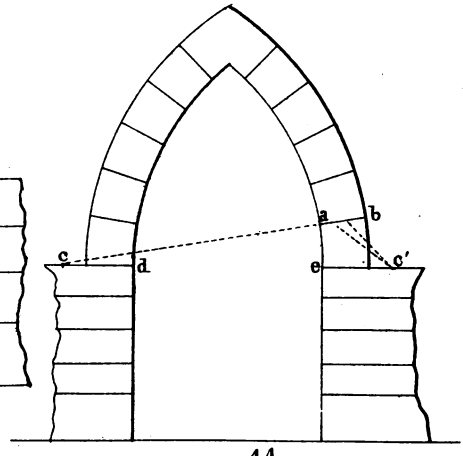

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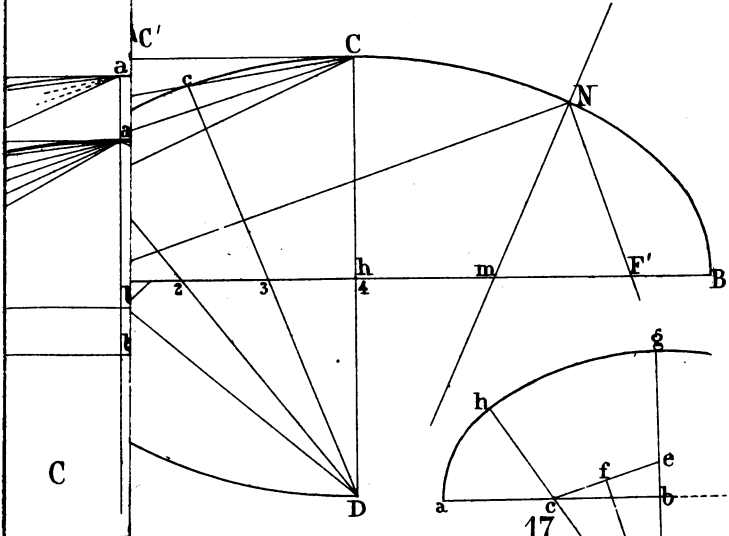

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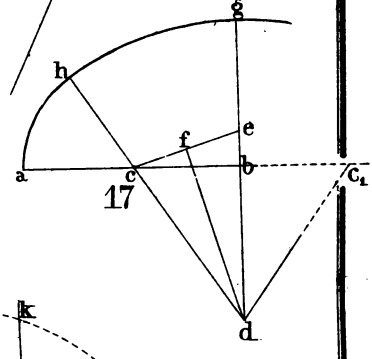

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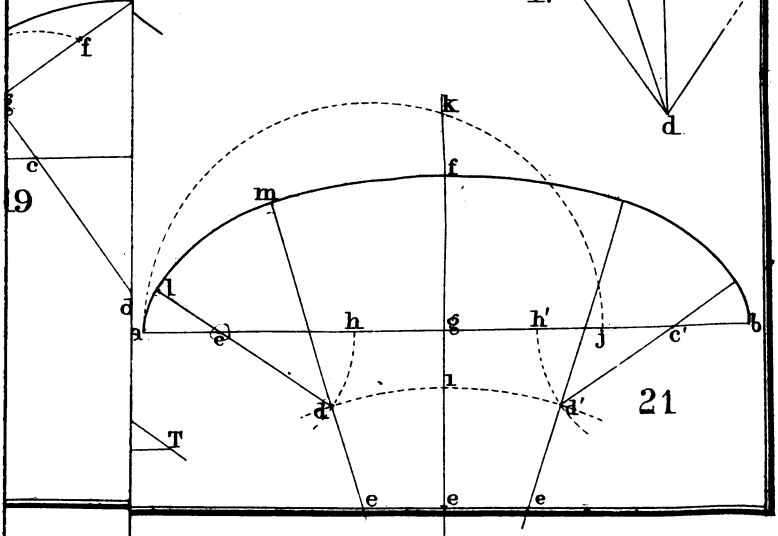

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similarly projected. The projections of the half face lines, EB and FD, would be elliptical arcs.[1]

II. *The directing Instruments.* — These are —

No. 1. The straight edge, applicable to any ruled surface, in the direction of its straight elements.

No. 2. The mason's square, applied wherever two lines, or two surfaces are to be perpendicular to each other.

No. 3. The pattern, MmZ_1oO, of the bed of a top stone of the abutment; also called an *impost stone*, or *springer*.

No. 4. The modification of No. 3, applicable to the bed, DdZ_4z, of the springer at the obtuse corner, D, of the abutment.

No. 5. The second modification of No. 3, for marking the bed, $MmZB$, of the springer at the acute corner, B, of the abutment.

No. 6. The bevel, $A_2'A'T$, for marking the joints Ii, Kk, etc., of the skew-back (19) in the plane, $U'O''A_1'$.

No. 7. The internal impost arch square, $D''B'1'$, for marking right sections of the intrados of the springers, in their proper positions relative to the face of the abutment.

No. 8. The external impost arch square, $18A_1'i'$, which determines right sections of the extrados, in case they are wrought of a cylindrical form, in their true relation to the level tops, as jYi, of the springers.

No. 9. The flexible pattern, $O_1O_4Q_1$, of the intrados, Π_2J, of a springer.

No. 10. The corresponding flexible pattern, $i_1Y_1a_1$, of the extrados, ii_2j, of a springer.

Nos. 11 and 12. The modifications of No. 10, which, put together, equal No. 10, and which apply to the partial extradosal surfaces of the two end springers; which exist in consequence of the different points, a_1 and a_2, at which the developed face line, B_2a_2 and helix ba_1, terminate.

No. 13. *The twisting frame*, 20, 21, 22 — 20′, 21′, 22′. This consists of three rulers, lying in three planes of right section, and, as shown by the drawing, coinciding with three elements,

[1] While writing these pages, an article on skew arches, by E. W. Hyde, C. E., has appeared in Van Nostrand's Magazine, Feb.-April, 1875; which may be read with much interest by those who have acquired a sufficient knowledge of Descriptive Geometry, of its applications to the problem as exemplified in the mainly graphical construction which I have here given; and of higher mathematics.

20 — 20′ ; 21 — 21′, and 22 — 22′, of a coursing helicoid, $C_0c_7S_2$. Their perpendicular distances apart *in a direction parallel to the axis*, are given in plan. Their angles with the horizontal plane are given in the elevation. From these data they can be rigidly framed together ; and can then be used to determine elements, in their true relative position upon a helicoidal side of a voussoir ; having first notched upon their edges their intersections with the helix C S_2. In order that No. 13 may be shifted along to determine successive elements, the perpendicular from 20 to 22 should be less than the length of the side of the stone taken in the direction of the axis, and hence called its *axial length.*

No. 14. *The soffit frame.* — This, used in case the intrados of a stone is wrought first, consists of three parallel pieces, 23, 24, 25, framed together in planes of right section and giving arcs of right section of the intrados, as R_2SPr''', of a stone. Their circular edges can be notched, by the aid of the drawings, so as to show their intersections with the inner helical edges, as R_3S, of the voussoir.

No. 15. The *arch square*, $e_33'e_4$, used in giving the cylindrical intrados of a stone from its helicoidal side when the latter is wrought first.

No. 16. A small, plane bevel, of the angle between a coursing joint and an arc of right section ; and *held against the curved arm* of No. 15, in the intrados, in order to guide No. 15 in a plane of right section.

No. 17. The helix templet, $b_4d_4c_4$, whose curved edge, though circular, will sensibly coincide with an inner helical arc, as B_0B_3, when placed against the helicoidal side of a stone. Making the perpendicular from d_4 to b_4c_4, equal to that from 7′, for example, to the chord of the two divisions, as 6′–8′, the elevation of an arc equal to b_4c_4, we have three points, b_4d_4 and c_4, to determine the required arc. No. 17 must be applied to give a helical edge as B_0B_3, before No. 15 can be applied, in succession to No. 13.

No. 18. A flexible pattern, $M_1M_4M_3B_4$, of the development of the intrados of a stone, gives the three remaining inner helical edges of that intrados, as B_3M_2, M_2M, and MB_0, after having found one, B_0B_3, by No. 17. Or, after beginning with No. 14, it will, by the aid of No. 16, give all these edges.

No. 19. Fig. 52, is a bevel frame for working the helicoidal

side of a stone from its cylindrical intrados, after having finished the latter by Nos. 1, 14, and 18.

No. 20. The *radial joint bevel*, $u_1c_4u_2$, is used jointly with No. 17, as indicated by the figure, to locate the corner edges as B_3b_3 of a stone upon the indefinite helicoidal side.

No. 21. The flexible pattern, $p_2p_3q_2q_3$, which replaces No. 18, for a voussoir which appears in the face of the arch. There must be a pattern of this kind for every voussoir of one end of the arch.

No. 22. This is a bevel varying for each stone in the face of the arch, and giving the angle between the tangent to the face joint, and the tangent to an inner coursing helix, at their intersection on the face-line.

We find this bevel for the point R'''R' as follows: The tangent, R'U', to the face-joint, meets the horizontal plane, A'B', at y'_2y_2. The tangent, R'Y'_2, to the coursing helix through R'''R' pierces the same plane at Y'_2Y_2, where Y_2 is thus found. R_1w_1, on the developed intrados, is the length of the vertical projection of the development of the portion, R''''R_1 — R'B', of this helix; hence making R'w' = R_1w_1, and projecting w' upon R_1w_1w at w, gives R'''w, the horizontal projection of R'w', the tangent to the helix at R'''R'. Then projecting Y'_2 upon R'''w, gives the trace Y_2 of this tangent upon A'B'. Hence Y_2y_2 is the like trace of the plane of the required angle, $Y_2R'''y_2$ — $Y'_2R'y'_2$. To find this angle we have only to revolve R'''R' about Y_2y_2 as an axis, into the plane A'B' at a point R_6, not shown; when $Y_2R_6y_2$ will be the required angle.

No. 22 is sometimes obtained mechanically, while building the arch, as follows, Fig. 7. Let ILF be a portion of the upper

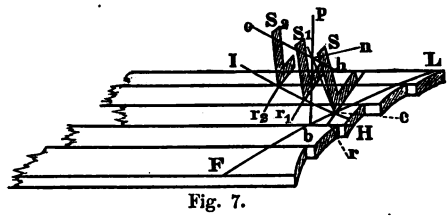


Fig. 7.

surface of the centring, coinciding therefore with the intrados of the arch, and on which the inner helices of both kinds, and the face-lines, are carefully chalked by means of thin flexible rules bent over the centring, and through points transferred upon chalked elements from the drawings.

Let FL be the face-line, and IH an inner coursing joint; and let r, r_1, r_2 be three arcs of right section. Then set up and fasten together three arch squares (No. 15) in the three parallel planes of r, r_1, and r_2; and tack on a strip oh, whose upper edge shall coincide with a coursing helix, by being equidistant from the angles of the squares, where it crosses their radial arms. Then pass a plumb-line, pb, along oh till, as at n, the bob rests on the face-line, when nc will be a face-joint, and Icn will be the required angle.

No. 23 is a pattern of the face of a face-stone, as FS_1B'', one end of which is in a face of the arch. It would be found, for each stone, on an elevation parallel to the face.

III. *The Application.* — This division of the problem will be sufficiently illustrated by the working of two of the principal stones; viz., one of the intermediate springers, and one of the voussoirs.

The springer, $IYJI_2i_2$ — $A'I_2'i'A_2'Y'10$.

This, being the most irregular stone, is further represented in oblique projection, Fig. 53. As in all other cases of irregular bodies, the stone is conceived to be inscribed in a rectangular prism, 14, 15, 16, 17 — 10, 11, 12, 13, from whose corners the points of the given body may be located by rectangular coordinates, which will therefore be seen in their true size in the oblique projection.

Merely adding that like points are lettered or numbered with like letters or numbers, the figure will mostly explain itself. Thus, ii_3, Fig. 53, $= ii_3$ from the plan; $i_3k = ii_2$ from the plan; and $ki_2 = 26i'$ from the elevation. Similarly for all other points, horizontal distances are taken from the plan, and vertical ones from the elevation.

1°. Bring its bed, ODZyo, Fig. 53, to a plane in the usual way, and mark its form by No. 3.

2°. On the edges given by No. 3, as directrices, work the lateral vertical faces, DZiI, ZyiY, etc., Fig. 53, square with the bed by No. 2; and thence the level portion, Yji, of the top.

3°. With No. 6, mark the joints, Jj and Ii, or test them.

4°. With No. 7, form channels of right section in the cylindrical face, II_2J; bring it to a cylindrical form by No. 1 applied in the direction of the elements, and mark its edges by No. 9.

5°. With No. 8, No. 1, and No. 10, similarly complete the cylindrical triangle, ii_2j, of the extrados.

6°. Work the helicoidal cheeks, Ii_2I_2 and $ijIJ$, which receive respectively a side and an end of adjoining voussoirs, simply by dividing the helical arcs, II_2 and ii_2 and JI_2 and ji_2, each into the same number of equal parts, and applying No. 1 upon the corresponding points of division, as shown in Fig. 53.

A voussoir. — As we face the vertical plane, one of these stones is, in detail, as follows: Mmb_1B_0 is its front end or *head*, forming a part of the transverse helicoid $MmLIi$. Its opposite and equal back end or *head*, is $M_2m_2b_3B_3$. Its right side or *bed*, MmM_2m_2, is a part of the coursing helicoidal surface, beginning at Mm. Its left hand *bed* is the equal surface $B_0b_1B_3b_3$. Then $MB_0M_2B_3$ is its cylindrical intrados, whose development is $M_1M_4M_3B_4$, and whose vertical projection is $B'3'$. Its similar extrados is $mb_1m_2b_3$, whose development is $mm_3b_5b_6$, and vertical projection $B_2'e_4$.

To form this stone from the rough block, choose a block in which the finished stone could be inscribed, then —

1°. By No. 13, form one of its indefinite helicoidal beds, as MmM_2m_2. In doing this, first cut three elements on which the edges of No. 13 will apply; then placing rule 20 on the *second* element, and rule 21 on the third element, rule 22 will be in a position to determine a *fourth* element of the same helicoid.

In the same way any number of other elements may be found, and the stone between them cut away by No. 1.

2°. By No. 17 and 20, placed together as shown on the plate, mark the helical inner edge MM_2, in its proper position relative to the determining elements marked by No. 13.

3°. By No. 15, having its straight arm applied to the elements given by No. 13, and its curved arm guided in a plane of right section by No. 16, applied on a small temporary plane area cut on the intrados, cut any desired number of arcs of right section in the intrados, $MB_0M_2B_3$; and finish the intrados as a cylindrical surface, by No. 1, placed in the direction of the elements, as shown in the plan.

4°. By No. 18, placed in the cylindrical intrados, first wrought, and with one of its edges coinciding with MM_2, mark the three remaining edges of the intrados. These edges will serve as directrices of the three remaining helicoidal surfaces of the stone.

5°. By No. 15, or, more exactly, by No. 19, Fig. 52, which consists of Nos. 1 and 15, very accurately made and firmly braced together, work the helicoidal bed, $B_0B_3b_3b_1$; and mark its corner edges, B_3b_3 and B_0b_1, by Nos. 17 and 20 together.

6°. Having now finished all the faces of the stone, except the extrados, which is to be left unwrought, and the ends, the latter can be wrought by No. 2, the square, from the intrados; one arm being held in the direction of the elements of the intrados, and the other applied to corresponding points of division of B_3M_2 and b_3m_2, transferred from the drawings.

In working a face-stone, as $R_3R'''r'''$, use No. 22 instead of No. 20, No. 21 instead of No. 18, and No. 23 in working the face.

Systems of Oblique Arch Stone-cutting.

105. ADHEMAR, with great elaborateness of graphical illustration and refinement of detail, devotes no less than eighty-three pages and seven large plates, to several different *methods* of cutting the stones of an oblique arch, arranged under *two systems*.

1*st*. *The method by squaring*, in which a rectangular prism of stone is first formed, in which the finished voussoir can be inscribed, and from whose sides, or edges, those of the required voussoir can be located with the square, by measurement.

2*d*. *The method by bevels*. This method is illustrated by the use of No. 2 (6 — 9), 15, and 19; a system which might be extended by framing Nos. 13 and 14 together, or by adding to No. 19 a second arch square, whose straight arm should be to the left of EE, giving the relative position of an *element*, and a *right section* of the intrados, and an element in each helicoidal bed, all in one frame, or compound bevel.

106. *Choice of circumscribing block*. This point, which may perplex a beginner, working only by the *first* of the above methods, may be settled in *three ways*.

First Method. The sides of the circumscribing rectangular prism may all be either parallel or perpendicular to the plane of right section. Thus, for the voussoir $R_3SR_6r_6$, the *length* of the prism (the prism not shown, to avoid confusion), would be the perpendicular between the radial elements at R_3 and R_6r_8; while its *width* and *thickness* would be shown in elevation by a rectangle circumscribing $5'5_1'2'm'$, since the vertical projections of r_7 and r_8 are $5_1'$ and m'.

This method is the most elementary, requiring, besides the square and straight edge, only Nos. 18 and 23; but it is obviously very *wasteful* of *labor and material.*

Second Method. Taking the same voussoir as before, project all its corners upon a plane tangent to the extrados along the element mid-way between r_6 and r_8. Then the least *rectangular prism* will be that, one of whose sides shall be in this tangent plane, and which shall include within it all the corners of the stone. By this method there will be very little waste of material in any case. And if this prism be first wrought, the voussoir can be simply extracted from it mostly by squaring from the edges of the prism, as before.

If, to save labor, the voussoir be wrought directly from the rough block, the latter will be chosen similar to the prism here described, but larger, and the method of cutting will be that above described in detail.

The price of labor and material will determine whether it will be best to cut the finished *prism*, or to risk an occasional spoiled voussoir mis-cut directly from the rough block, or from a block which by an error of choice may be too small.

Third Method. Here the faces of the circumscribing prism are all parallel or perpendicular to the plane of the face of the arch. This method is particularly adapted to a voussoir one end of which is in the face of the arch, and can therefore be immediately marked upon the face of the finished provisional circumscribing prism.

107. *Adaptation to a cut stone spandril.* In this case, the tops of the face-stone are finished with vertical and horizontal surfaces, as in Pl. IV., Fig. 30, and extending through the thickness of the surmounting parapet. The *horizontal surfaces are plane*, and the vertical ones are cylindrical, having the extradosal coursing helices for their directrices. The cutting of the face-voussoirs thus designed, offers no special difficulty.

Useful Numerical Data.

108. By some writers the problem of the oblique arch has been treated almost wholly by the equations of its lines, and of their projections; from which all its points could be located by merely plotting to scale the results of computations, made by substituting numerical data in the formulas.[1]

[1] *Bashforth*, London, 1855. *Buck*, London, 1839. *Graef*, Paris, 1853.

But such treatment is less adapted for general use than the graphical one, as most thoroughly exhibited by *Adhemar;* and which is all-sufficient, mainly on account of the *simplicity and similarity* of the principal lines, on which all the points depend. Nevertheless, a few simple formulas, such as all can use, are useful as checks upon purely graphical constructions, especially when the latter are of half, or whole size, upon a platform, as they should be in practice.

109. Such formulas are the following, with the numerical data, and results, used in the present example, substituted: —

The half span $O'''B' = S = 6.5$ ft.

$O'''B'O'' = a = 30°$.

Then $O'''O'' = S. \tan a = 3.75$ ft.

and $O''B' = R = \sqrt{S^2 + O'''O''^2} = 7.5$ ft. (7.504 ft.)

Again, let the angle of skew, $EAC, = \theta$

then $EAG = 90° - \theta$,

whence $BG = 2S. \tan (90° - \theta)$ (1)

and $AB = \sqrt{AG^2 + BG^2} = \sqrt{13^2 + (2S. \tan (90 - \theta)^2}$

or, $AB = 2S. \sec. (90 - \theta) = 2S. \operatorname{cosec} \theta = \frac{2S}{\sin \theta}$.

Also $CC_1 = A'E'B' = l = R \times .0174533 \times 120°$ (2)
$= 15.708$ ft.;

where $A'E'B' = 120°$,

and .0174533 is the length of 1° where $R = 1°$.

Likewise, putting $O''B_2' = R_1$,

$n_2 a_2 = A_2'F'B_2' = l_1 = R_1 \times .0174533 \times 120°$, (3)

where $R_1 = 7.5 + 2.5 = 10$, whence $n_2 a_2 = 20.94$ ft.

Further; let $C_2 y_4$, perpendicular to AB, $= L = 14$ ft.

Then, as $A_2C_2y_4 = (90 - \theta.)$

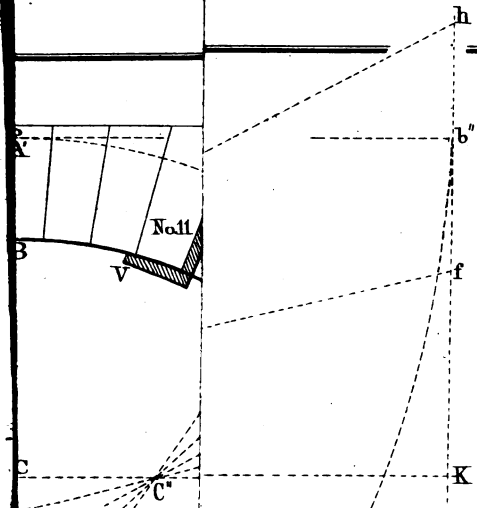
$AC = B_1D_1 = A_2C_2 = L. \sec (90 - \theta) = L. \operatorname{cosec} \theta = \frac{L}{\sin \theta}$

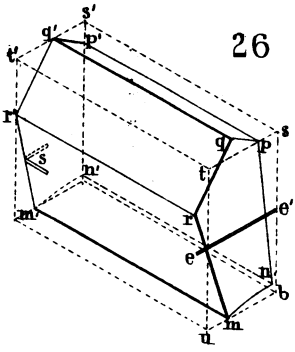
And, see (1) and (2), $AB_1 = CD_1 = \sqrt{AG_1^2 + B_1G_1^2}$ (4)

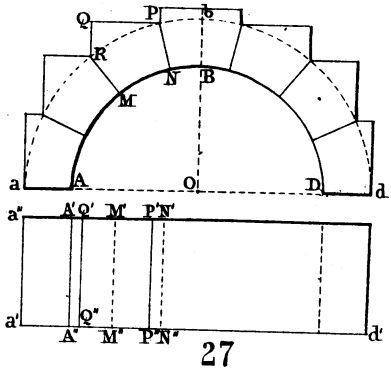
Now suppose CD_1 to be divided into n equal parts of which D_1IV are m parts. Then $D_1IV = \frac{m}{n} CD_1$.

or, in this example, see (4), $D_1IV = \frac{4}{5}. CD_1$.

Let $AB_1G_1 = B_1D_1C = \beta$, then $\frac{AG_1}{B_1G_1} = \tan \beta$. Or, as by (1) $B_1G_1 = BG = 2S. \cot \theta$







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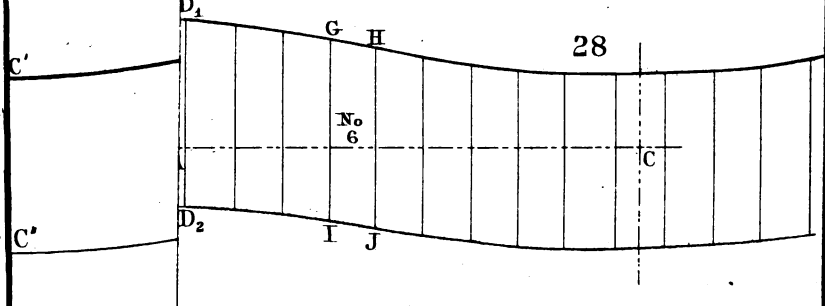

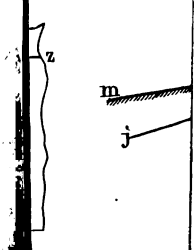

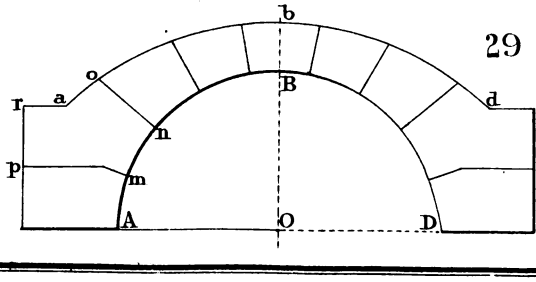

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$$\frac{AG_1}{2S.\ \cot\theta} = \tan\beta.$$

In the triangle B_1D_{1IV}, we now know two sides and the included angle β, whence the angle D_1B_{1IV}, or γ, and thence its complement $N_1B_1b_6$, can be readily found.

But B_1b_6 is already known, as $= CC_1$, and calling N_2, the point where the developed spiral B_1N_1 meets the springing line AC produced, we have, in the right-angled triangle $N_2B_1b_6$, $N_2b_6 = CC_1 \tan(90-\gamma) =$, in this example, one third of the pitch, P, of a coursing spiral, of either the intrados or extrados.

Comparing the triangles, $O''O'''B'$ and $O''o'''B_2'$ and calling $o'''B_2' = s$, we have

$$R : O'''B' :: R_1 : s.$$

$$\text{whence } s = \frac{R_1}{R}.\ S,$$

$$\text{and } A_2n_9 = 2s = 2\,\frac{R_1}{R}.\ S.$$

$$\text{from which } B_2n_9 = 2s.\ \tan(90-\theta) \qquad (5)$$

$$\text{and } A_2B_2 = \sqrt{(2s)^2 + B_2n_9^2}$$

Then, calling $a_4ba_1 = \delta = B_2dc_1$.

$$\text{Tan } \delta = \frac{l_1}{BG} \text{ (see (1) and (3))}$$

$$\text{and } ba_1 = \sqrt{BG^2 + l_1^2} \text{ or, } = \frac{BG}{\sin\gamma}.$$

Such simple calculations as these will form an ample check on any graphical errors.

Modifications of the Orthogonal and the Helicoidal Systems.

110. Among these are, *first*, the following *minor* ones: —

1°. Making the heads of the voussoirs plane instead of helicoidal; which may be done without sensible error in large arches.

2°. The substitution of vertical *plane* surfaces for vertical cylindrical ones (107) in the spandril steps of large arches.

3°. The substitution, always unadvisable, of separate hyperbolic paraboloids, for the helicoidal *beds* of the voussoirs; as in taking B_3b_3 and B_0b_1 as the directrices of a hyperbolic paraboloid to be generated by a straight line moving upon them so as to divide them proportionally.

4°. *Avoiding the acute diedral angles*, between the face and the intrados, near the *acute* angles, as B, of the abutments, by terminating the *cylindrical* intrados by a plane parallel, and near to AB; through r_5 for example, and then completing the intrados between this plane and AB by a *conoid*, whose *directrices* should be the ellipse in the plane r_5, and a vertical line on BD, as at Q, and equal to O'''E'; and whose *plane director* should be horizontal.

5°. The substitution of brick for stone; either wholly, or except in the face ring of stones.

6°. The substitution of a circular for an elliptic face. This will make the right section an arc of an ellipse whose longer axis will be vertical.[1]

111. *Among much more important kinds* of *modification*, are the following: —

1°. *Convergent arches.* — When an arch is quite long, the central portion of it may properly, and with great economy, be built as a right arch, either of brick or stone. Then, near the ends the coursing joints may gradually bend till perpendicular to the face line at their intersections with it.

2°. *Hart's System.*[2] — In this system, the *transverse joint surfaces* are *planes*, parallel to *the face;* while each *coursing joint surface* is a *conoid*, generated by a line moving parallel to the face as a plane director, and upon the axis, and a helical coursing joint as *directrices*.

3°. *Cylindrical coursing joint surfaces.* — This very interesting modification of the orthogonal or equilibrated system, is strongly advocated by *Adhemar*,[3] *first*, as dispensing with all cut surfaces except plane and cylindrical ones, so related as to be the ones most easily wrought with economy of material, labor, and graphical construction; *second*, as wholly avoiding those components of pressure which act towards the faces, and tend to produce dislocation.

The transverse joint surfaces in this system are planes parallel to the face.

The coursing joint surfaces are the horizontal cylinders, per-

[1] Such arches, on the helicoidal system, are the subject of a work by *Praly*, Paris, 1853.

[2] *John Hart*, London, 1848.

[3] Only regretting that prescribed limits imperatively forbid adequate illustration of this system; the reader is referred to the ample exhibition of it given by *Adhemar*, Paris, 1861.

pendicular to the plane of the face, which project the trajectories, or intradosal coursing joints of the orthogonal arch, Prob. XII., upon the plane of the face.

4°. *The Cow's Horn.* — The form of oblique arch whose intrados is a portion of the warped surface so named, may be substituted, in case of small arches, as oblique door-ways. See Pl. VII., Fig. 54.

The Cow's Horn (Corne de Vache) is a warped surface, of the kind having three given directrices. In its usual form, it is generated by a straight line, AC — A′C′, moving so as always to rest upon two equal and parallel semicircles, AB — A′N′B′, and CD — C′N′D′, whose diameters are in the same plane; and upon a straight line, OO″ — O′, perpendicular to the planes of these semicircles, at the point of symmetry O′.

This surface is the *intrados* of the arch. The semicircles are its face lines. The semicircle on L′M′ is the ideal extrados of construction. The face-joints, as $a'c'$ and $m'p'$, are in pairs, symmetrical with O′N′; the coursing joints, ab — $a'b'$, etc., are elements of the warped surface; and the available height of the passage is O′N′. This height diminishes with the increasing obliquity of the arch, till, when A′ and D′ coincide, the arch degenerates into two cones, tangent to each other on OO″ — O′ and the passage becomes closed. This is an objection to this form of arch.

EXAMPLES. — 1°. Make a full construction, with patterns, etc., for the Cow's Horn.

2°. Do. of a segmental, helicoidal arch, *left*-handed, that of Prob. XIII. being *right*-handed, of 12 ft. span, 55° skew, 120° arc of right section and 2 ft. radial thickness, and 12′ : 6″ perpendicular between faces.

3°. Do. of a full centred, helicoidal arch, with 60° angle or skew.

4°. Construct an arch on the cylindric system.

5°. On Hart's system.

6°. A segmental, helicoidal, left-handed arch of 14 ft. span; perpendicular length 12 ft., 48° skew; 135° arc of right section; radial thickness 2 ft., 2 in., and with the face voussoirs adapted to a cut stone spandril wall. Add several oblique and isometric projections of springers and voussoirs.

Wing-Walls.

112. Prismatic and pyramidal wing-walls have been sufficiently illustrated in outline on Pl. I., Figs. 6–9. The vertical, quarter-cylindrical one, with an oblique plane top as in Pl. I., Fig. 8, is described in my "Elementary Projection Drawing." The top of the straight wall, which contains the face of the

arch, or passage, and which is flanked by the wing-wall, is sometimes horizontal; while either wing-wall is a segment of a cylinder, cut off obliquely, as in Pl. I., Fig. 9. The top of either wing-wall is then a portion of a right helicoid, having a vertical axis; that is, if such a wing-wall were a quarter-cylinder, its coping would be one quarter of a revolution of a square-threaded screw of stone.

All the foregoing forms are in use in the wing-walls of small arches, culverts, and mounded cemetery vaults.

There yet remain, however, two forms of wing-wall, adapted to larger structures, and which will now be described.

113. Pl. VIII., Fig. 55, is a sketch, in plan and elevation, of the crossing of two lines of communication, at different levels, and at an acute angle. The lower one, *rr*, passes, by an oblique arch, flanked by wing-walls, AC and BC_1, through the embankment, MM, which supports the upper line. The axis, parallel to *rn*, of the arch, makes an angle, C*nr*, with the direction, CC_1, of the embankment through which the arch runs.

This understood, it is determined that the wing-walls shall extend a given distance in front of CC_1, without being as far apart at A and B as they would be if wholly curved. They are, therefore, in plan, made partly straight and partly curved, with a quite small radius, as O*o*.

The straight wall, containing one face of the arch, lies between the vertical planes CO and C_1O_1. Then, if the face of the wall slopes, that of the curved portion of each wing-wall will be a segment of a cone having a vertical axis; and the faces, both of the arch-wall and of the straight part of the wing-wall, will be tangent planes to this cone. All parts of the face of the wall will thus have the same batter.

From the complete plan, it will be seen that *if* the straight parts of the two wing-walls are to be parallel to the centre line of the arch, and equidistant from it, the conical portions must be *unequal segments*, and of unequal radii, the larger segment, adjacent to C_1, having the less radius.

PROBLEM XIV.

The compound, or plano-conical wing-wall.

I. *The Projections.* 1°. *Preliminaries.* — Pl. VIII., Fig. 56, represents the plan and elevation of a wing-wall, similar to that part of Fig. 55 which is to the *left* of CO; but with the addition of a pier at the foot of the wall, and a coping; also, the plane V, instead of being taken as in Fig. 55, parallel to CC_1, is parallel to a plane, as OC, perpendicular to CC_1.

Let the *vertical* line at O be the axis of the inverted cone of revolution, a part of whose convex surface is the face of the conical part of the wall.

The circle with radius, OA, of four feet, is the horizontal trace of this cone, and the arc AG, of 60°, is that of the conical portion of the wall. The tangents to AG at A and G are respectively the horizontal traces of the planes of the faces of the arch-wall, and of the straight portion of the wing-wall; which are respectively tangent to the conical part, along the elements whose horizontal projections are VB and VGT. Let the height of the wall under the coping be 6 ft. 4 in. = 76 ins. and the batter, $\frac{1}{4}$. The horizontal, AB, of the batter is, then, 19 ins.; and, making BC, the thickness of the wall at top, 2 ft., we reach, at C, the back of the wall.

2°. *Outlines of the elevation.* — We must now turn to the elevation, and make D′C′ = 6 ft. 4 ins.; draw C′B′ parallel to the ground line RD′, and project A at A′ and B at B′. Then A′B′C′D′ is the section of the wing-wall at its junction with the arch-wall, in the vertical plane OC. Hence V′, the vertical projection of the vertex of the conical portion of the face of the wall, is found by producing the element BA—B′A′, parallel to V, till it meets the axis O′O.

The section of the coping, in the vertical plane OC, is C′E′F′H′; 8 inches thick by 2 ft. 3 inches wide.

Next, suppose the plane slope, QC, of the embankment, behind the wing-wall, to be one of $1\frac{1}{2}$ to 1, and to contain the back top edge, C_1C—C′, of the arch-wall. Accordingly, make D′Q = $\frac{3}{2}$ D′C′, then, as the plane of the slope of the embankment is perpendicular to the plane OC, Figs. 55 and 56, its horizontal trace, as AB, Fig. 55, PQ, Fig. 56, is parallel to CC_1. Hence PQC′, Fig. 56, is the plane of this slope.

3°. *Conditions for the design of the top of the wall.* — Let these be three, as follows: —

1*st.* Its front and back *edges* shall be *plane* curves; sections of the front and back *surfaces* of the wall.

2*d.* It shall be of uniform width, as seen in horizontal projection.

3*d.* Its *straight* elements shall be horizontal.

Let the plane SRB′, parallel to PQC′, be that of the front top edge, whose vertical projection is therefore in RB′. To find its horizontal projection, *a*TB, proceed as in Des. Geom., Prob. LVII. Thus, projecting G at G′, gives VGT—V′G′T′ as one of the limiting elements of the conical face of the wall, and T′, its intersection with RB′, is the vertical projection of a point of the required edge, whose horizontal projection, T, is therefore the projection of T′ upon VG produced. In like manner, any other points can be found. The front upper edge, TB–T′B′, of the conical part, is an elliptical arc, whose horizontal projection, TB, is also elliptical, but in this case so slightly so, that it may be represented, without sensible error, by a circular arc whose centre, *k*, on OA, may be found by trial.

The front edge, *a*T, of the straight portion of the wall, is tangent at TT′ to TB—T′B′, it being the intersection of the tangent plane, GS, to the conical part, along the element GT, with the plane, SRB′, of the top edge. Hence (Des. Geom. 172) S is a point of this tangent, which is therefore projected in ST—R′T′.

Supposing the wall to be terminated at *a*′, at a height, *f*′*f*″, of 20 ins. from the ground; project *a*′ upon ST at *a*, make *af* = BC = 2 ft., by condition 2°; project *f* upon the horizontal *a*′*f*′ at *f*′, and *f*′C′ will be the vertical projection of the *back* top edge of the *wall*, by condition 1°. By condition 2°, draw CK with centre *k* (in general, a curve at a constant normal distance from BT), and limited by *k*K, and K*f* will be parallel to *a*T.

Now, by condition 3°, draw T′*h*′ parallel to the ground line, till it meets *f*′C′ at *h*′; project *h*′ at *h* upon the curve KC, and T*h*—T′*h*′ will be one element of the top of the wall. Again: project K at K″, draw K″*l*″, project *l*″ (which is on *a*′B′) at *l* on *a*T; and K*l*—K″*l*″ will be another element of the top of the wall; which, as can now be seen, consists of three different warped surfaces, as follows.

By condition 3°, all have the plane H as their common plane director. Then, —

1*st*. K*f*—K″*f*′ and *al*–*a*′*l*′, both being straight, are directrices of *af*lK—*a*′*f*′*l*′K″, which is thus a *hyperbolic paraboloid* (Des. Geom. 278).

2*d*. K*h*—K″*h*′ being curved, while *l*T—*l*′T′ is straight, these are the directrices of a small conoidal portion, K*h*T*l*—K″*h*′T′*l*′. (See 114, p. 92.)

3*d*. T*h*BC—T′*h*′B′C′ is a warped surface without specific name, of the kind having two directrices, TB—T′B′ and C*h*—C′*h*′ and a plane director, which is H.

These surfaces are evidently tangent to each other along K*l*—K″*l*′ and T*h*—T′*h*′.

4°. *The pier.* — This is rectangular in plan, *fe*, being one side, and perpendiculars to it at *e* and *f*, 3 ins., longer than *ef*, being the adjacent sides; so that the coping stone, *odg*, of the pier shall be a square of 2 ft. 6 ins., overhanging on three sides. The top of this stone is finished as a square pyramid 8 ins. high.

The pier having no batter, project *e* at *e*′; and the vertical at *e*′ is a corner of the pier, and *e*′*a*′ is the intersection of its back with the face of the wall. Project *d* in the vertical at *d*′, for the corner of the pier; and *c* in the vertical, *c*′*c*″, for the junction of the pier coping with the front of the wall coping; both being vertical surfaces. Then *c*′F′, and the parallel through H′, are the upper and lower front edges of the wall coping.

Coping edges. — FL*c*—*c*″H′, being the intersection of the warped top surface of the wall, produced, by a vertical plane and cylinder, whose combined horizontal trace is FL*c*, is not strictly a plane curve; but it is so near to the plane curve, BT*a*—B′*a*′, that it may be considered as one without sensible error; especially in practical cases, where OA would be several times 4 ft., and hence the changes of form of all the curved surfaces much less quick. F′L′*c*′ would be strictly constructed by projecting points of FL*c* upon the vertical projections of the same horizontal elements, K*l*—K″*l*′, etc., on which they were taken. The other edges of the coping are FL*c*—F′*c*′ and CK*f*—E′*f*″.

5°. *The wall joints.* — In the *face* of the wall, the *coursing joints* of the conical part are horizontal circles, as *b*′*r*′; right sections of this portion. Those of the straight part are parallel to G*e*. The *heading joints* are elements of the conic portion, and *parallel to* GT on the straight portion.

The bed surfaces. — To avoid acute-angled edges of stones by making adjacent surfaces mutually perpendicular, the bed sur-

faces, that is, those extending from AG, *br*, etc., to the back of the wall, may be formed in two ways.

First. If two lines, as V′B′ and $v'j'j''$, perpendicular to each other, as at j', revolve about the common vertical axis V—V′v', which they both intersect, V′B′ will generate the conical *front* of the wall; $v'j'j''$ will generate a conic *bed* surface, whose vertex is Vv', and everywhere normal to the conical front of the wall, and jj' will describe the horizontal circle, jp—$j'p'$, intersection of these two cones, as desired for a coursing joint. With like results, revolve $b'b''$, and the other parallels to $v'j''$, about the same axis, V—V′v'.

Second. If VV′ be made the centre of a series of spheres, of radii V′A′, V′b', Vj', etc., each of them will be normal to the conical face of the wall, at their intersections with it, since the elements of the wall are radii of the spheres; also they will intersect the conic face in horizontal circles, as before.

Preferring the former system as simpler, let the *beds* of the stones of the conic wall be conic surfaces generated by $v'j'j''$, etc., and let the *ends* of the stones be in vertical planes, VB, VI, etc., radiating from the axis V–V′v' of the conical wall.

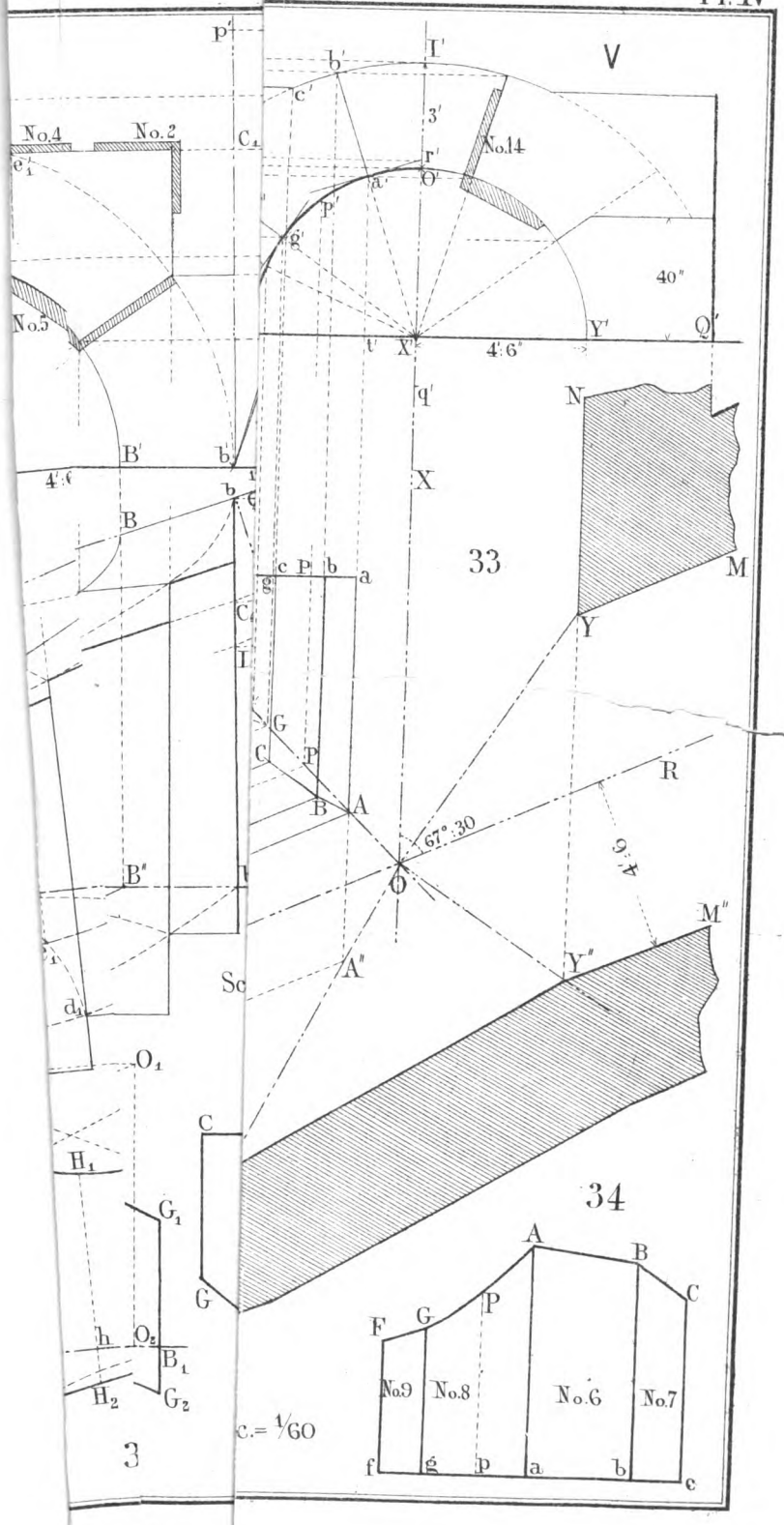
Thus, *sqrp*IJ and $s'q'r'p'$I′J′U′W′ are the complete projections of one stone of the conical wall.

The *beds* of the stones in the straight wall are planes, tangent, as on r'W′ and p'J′, to the conical beds in the conical wall. Their *heads* are in vertical planes parallel to VGT.

The coping joints. — These simply divide the top of the coping suitably, so as to break joints with the upper stones of the wall, and are in vertical places through the horizontal straight elements KL — K′L′, MN — M′N′, etc., of the top of the coping.

Thus, MN — M′N′P′Q′, LK — L′K′K″L″, etc., are vertical joint surfaces of the coping.

6°. *Location and limitation of the face joints.* — It is desirable that a coursing joint should coincide with the top of the pier, but not always convenient to make a suitable common divisor of the heights of the pier and wall; hence arises the problem: to arrange the stones in courses of diminishing thickness upwards, and so that a coursing joint shall coincide with the top of the pier. Now *parallel* lines will be divided proportionally by lines which meet at a point, P; hence, if the latter lines divide a given line, A, equally, they will divide a line, B, *not* parallel to A, unequally.


V
No.4
No.2
No.3
No.14
40"
4'.6"
33
34
67°.30
4'.6
No.9
No.8
No.6
No.7
c.= 1/60
3

Hence, choosing in this example *five courses* of stones, assume any point as a' on the level $a'b'$ of the pier, drop a perpendicular $B'B''$ from B', and draw $a'B'$ and $a'B''$. This done, adjust any convenient edge scale of equal parts, by trial, till the 0 of the scale being placed on $a'B''$, the point 1 of the scale shall fall on $a'b'$, and 5 of the scale on $a'B'$. Holding the scale in this position, prick off the other points of division from 0 to 5 upon the paper, and draw lines from a' through the points so found, to meet $B'B''$, as at y, etc. Then, through y, etc., draw the horizontal joints.

To avoid acute intersections of the coursing joints, $q'p'$, etc., with the top edge, $a'B'$, of the wall, the former must, as shown, terminate on convenient heading joints, as $b'a'$ on $u'x'$, $q'p'$ on $G'T'$, etc.

II. *The Directing Instruments.* These, besides Nos. 1 and 2, as in all other problems, are as follows: —

No. 3, a bevel, set to the diedral angle made by the slant height of the pyramid, *odg*, with the vertical surfaces of the pier coping. This angle is here shown by revolving the vertical plane of this bevel till horizontal, as at $o''f_1a$.

No. 4 is a pattern of each face, as *odg*, of the pyramid, found by revolving it about *dg* till horizontal, giving $o'''dg$, where $o'''f_1 = o''f_1$.

No. 5 is the twisting frame for working the top of the coping of the straight wall. It consists of two rulers framed together so as to be in parallel planes, and so as to give the relative position of the edges, as $Zc - Z'c'$ and $Yf - Y'f''$, of the front and back of this coping. These rulers are shown in their real relative position, in the separate figure (No. 5), by making $c3$ and $c2$ horizontal and equal respectively to cZ and fY on the plan; and then $3z$ and $2Y$, perpendicular to $c3$ and $c2$, and respectively equal to $3Z'$ and $2Y'$ on the elevation. The perpendicular distance of the rule cYY'' behind czz'' is af; so that the former applies to the edge $Yf - Y'f''$ of the stone, while the latter applies to $Zc - Z'c'$.

A similar frame can be made for the next stone, YL, of the coping.

No. 6 is an elliptical templet fitted to the front top edge of the conic wall. It is found simply by revolving the elliptic arc, $BT - B'T'$, around its transverse axis, $BO - B'a'$, till

parallel to the plane V, when ordinates, as $N''N_1$, to this axis, will appear as at $n'''N_2$, perpendicular to $a'B'$, and in their real size, since $N''N_1$ is horizontal, and therefore seen in its real size.

No. 7, not shown, is similar to No. 6, and similarly found, and shows the real form of the curves, as $FL_1 - F'L_1'$, of the front of the coping, assumed as plane curves because so near to B'T' (4°). Hence, the ordinates for finding No. 7 will be laid off on perpendiculars to $F'c'$, and equal to their horizontal projections, which are perpendiculars from FL_1 to BO.

No. 8, the *dip bevel*, shows the inclination of the elements, $b'b''$, etc., of the conical beds of the stones to the horizontal plane.

No. 9, the *coursing joint templet*, is one of a set, one for each coursing joint being evidently required by the increasing radii of the latter. Its acting edge, *ism*, is cut to the circular arc of a coursing joint.

No. 10 is flexible, and gives the relative position of a radial joint, as $j''j'$, and a coursing joint, as $j'p'$; and hence, when laid flat, gives the angle between an *element* and an arc of the *developed right section* of a bed cone.

There is thus one for each bed, applicable to the two conic surfaces, top of one stone and bottom of the next above it, which unite on that bed.

Thus, in the one shown, the curved edge of radius $v'j'$ is an arc of the development of the horizontal circular section, $jp - j'p'$, of the bed cone whose vertex is v', while its straight edge coincides with $Oj - v'j'$, produced.

No. 11 shows in like manner the relative position of a *heading* and a *coursing* face joint, as $s'q'$ and $p'q'$; hence, when laid flat, its curved edge, in the example shown, is drawn with the radius $V'j'$, slant height of the face cone to $p'j'$, as a base or right section, while its straight arm coincides with the element $V'j'$.

No. 12 is a twisting frame for the curved portion of the coping. It consists of three or more rulers, framed together, each in a vertical plane, and so that their acting edges have the relative position of three or more elements, as at $X7 - 7'$; $L_1K_1 - L_1'K_1'$, and $LK - L'K'$, of the top of the coping.

No. 13, used in connection with No. 8, as indicated, gives the vertical joints, as $L'L''$, of the front of the coping.

No. 14, a bevel between the bed and back of a stone, serves to keep these surfaces in their proper relative position, so far as the back, being exposed, is wrought.

III. *The Application.* This, so far as not already sufficiently understood from the description of the directing instruments, can be sufficiently illustrated in the working of a stone from the body of the conical wall, and of the top of the wall as a whole.

Take for illustration the stone $rspq\text{IJ} - r's'p'q'\text{U}'\text{J}'$; but as it is obscurely represented in the general projections, see Fig. 57, an oblique projection of it, which explains itself by lettering like points with the same letters as in Fig. 56.

As in all other cases, points, as *ii*, not in the edges or faces of the auxiliary circumscribing rectangular prism $r_1ur_s - r_2\text{U}'\text{I}''$, are found by one or more ordinates, perpendicular to those faces, and to each other; and hence seen in their real size in Fig. 57, where all parallels to r_1r_s, r_1p'', and r_1u, are shown in their real size.

Thus, in the two figures $r_1i_2 = r_1i_2$; $i_2i_1 = i_2i_1$ (plan) and $i_1i' = i_1i'$ (elevation).

Having then chosen a rough block capable of containing the finished stone, proceed as follows: —

1°. Work a portion as mm_1nn_1 of its upper surface to a plane, and on this *temporary* plane portion (sometimes called a *surface of operation*) mark the circular front edge, *pq*, by No. 9.

2°. By No. 8, having its straight arm in the temporary plane, *mpnq*, and radial to the curve, as shown, form any number of elements of the conical top, $p\text{J}Iq$, of the stone; and gauge the thicknesses, as *q*I, etc., on these elements, Fig. 56, which will give the back edge, IJ, of the stone.

3°. By No. 10, placed as shown, mark the *radial* edges, *q*I and *p*J, of the *ends* of the stone.

4°. Work the conical front, square with the top, by No. 2, the square, as shown, placing its arms to coincide with *elements* of both surfaces, and mark the vertical joints, *sq* and *rp*, by No. 11, placed as shown.

5°. The bottom edge, *rs*, is equidistant from *pq*, measured on elements, or it may be marked by one of the set No. 9, or by one of No. 11, or yet again by No. 15 (not shown, but readily found) the development of the entire front face, *rspq*—

$r's'p'q'$ considered as on the convex surface of the cone, VV', of the face of the conical wall.

6°. The ends are wrought simply by the straight-edge, from their edges, already found, as directrices. The bottom bed is square with the front, and also tested by the suitable form of the set, No. 10.

7°. *The top of the wall.* — Having finished the body of the wall, except the top, its front edge, B'Ta — B'a', can be immediately located by measuring on the heading joints, from any one coursing joint, as $a'b'$, or from a firm platform built at any suitable level, as a horizontal reference plane. Having thus the edge, B'a', as a directrix, any number of level chisel lines can be cut in the top of the wall, by the aid of a spirit-level, till the top of the wall is finished.

Then, having wrought the *top of the coping*, by the forms of Nos. 5 and 12, suited to each stone, gauge the coping everywhere of a uniform *vertical* thickness, which may be done by forms of No. 13, suited to different positions along FLc. The under surface may then be wrought, either by suitable forms of Nos. 5 and 12, or even by No. 1 only.

When material and labor are cheap enough, the somewhat difficult stones of the coping of the conical wall might be wrought by the *method of squaring* (105) from a completely finished circumscribing rectangular prism.

EXAMPLES. — 1. Construct a front elevation of the wing-wall.

2. Construct an *isometric* drawing of a stone from the body of the conic wall.

3. Construct an isometric view, and an oblique projection of a coping stone, to illustrate the cutting of it by squaring.

4. Work out the entire problem for the opposite wing-wall, C_1B, Fig. 55.

5. Work out the problem of the vertical quarter cylindrical wing-wall with a right helicoidal top (whose elements will therefore be horizontal).

The Conoid.

114. *The conoid* is a warped surface, which is generated by a straight line which moves upon a straight line and a curve as *directrices*, and always parallel to a given plane, called its *plane-director*.

In its simplest form, partly shown in Pl. VIII., Fig. 58, which is an oblique projection, the directrices are a straight line, OF, and a circle as that on AC, both perpendicular to the plane director, RCO, the plane, FOQ, being also perpendicular to that of the circle. The plane RCO is **H**, and RBC is **V**.

115. *Elliptic sections.* — Every plane section parallel to ABC is an ellipse. Let *atb* indicate such a section. Since $te = dq$, and $TE = DQ$, we have from this, and the triangle OAQ, $TE : AQ :: te : aq$, which expresses a property of two ellipses having an axis in each, equal. Hence *atb* is an ellipse.

116. *Tangents.* — If a line moves upon three fixed lines, it can have but one position at each point of any one of these lines. But a hyperbolic paraboloid consists of two sets of elements, *all* those of each set parallel to one plane, and *each* element of each set intersecting *all;* and hence, *any three* of the other set. Hence, if we take any three lines, as RK, *rk*, and OF, tangent at points on the same element, as TH, of the conoid, and parallel to one plane, **V**, they will be elements of one generation of a hyperbolic paraboloid, whose other generation is formed by moving TH, upon these tangents. TH being parallel to **H**, will remain so, and hence R*r*O is one of its positions, and KF another. That is $KQ = FO$. Thus RTKFHO is a hyperbolic paraboloid, tangent to the conoid along TH, and having **H** and **V** for the plane directors of its two generations.

117. *Normal surface.* — But as we could, in the last article, have taken any other tangent at T, and parallels to it at all points of TH, there may be an indefinite number of tangent hyperbolic paraboloids along TH. Of these, one will contain all those tangents which are *perpendicular* to TH. Now, let all these latter tangents be revolved 90°, about TH as an axis, and they will all become normals to the conoid on TH. But as they do not thus change their position relative to each other, they will still form a hyperbolic paraboloid, which is thus the *normal surface* along a given element.

118. *General conclusion.* — No distinctive property of the *conoid*, but only those of the hyperbolic paraboloid, having been employed in this demonstration, this shows that the result is general; viz., that the *normal surface* to any warped surface at a given element, is a *hyperbolic paraboloid.*

PROBLEM XV.

The conoidal wing-wall.

I. *The Projections.* — This novel form of wing-wall is founded in the idea, that, at the foot, FC, of the wall, where the pressure of earth from behind is slight, no batter would be necessary to give increased stability by increased thickness at the base; while, on the other hand, at AB — A′B′, where the height is greatest, the need of a slope or batter to the front would also be greatest.

1°. *To fulfill these conditions*, let the face of the wall be a quarter of a right conoid having the quadrant EF — E′F′ for its curved direction; the perpendicular, OF — O′, to the plane V, for its straight directrix (O′ being the intersection of FO with B′E′, produced, where B′E′ has a slope of 3 to 10), and the plane V for its plane director.

Hence, divide E*h*F, a quadrant of 9 ft. radius, into any convenient number of parts, and OE, O_1g, O_2h, etc., drawn through the points of division, and parallel to the ground-line D′F′, will be the horizontal projection of elements of the conoidal face of the wall.

2°. *The vertical projections of these elements* will be found by projecting E at E′, *g* at *g′*, etc., and drawing E′O′ with a batter, E′*b′*, of 3 to 10, then *g′*O′, etc. But, in this figure, O′ is thus made inaccessible; hence proceed as follows: —

Draw A′B′ at the intended height, 10 ft., of the wall, and produce it as the vertical trace, A′*e′*, of a horizontal plane, in which is the quadrant B*ae*, whose centre is O. Divide this quadrant in the same manner as EF, as at *a*, *c*, etc., project *a*, *c*, etc., at *a′*, *c′*, etc., and as *a′g′*, *c′h′*, etc., necessarily pass through O′ (being identical with the vertical projections of the elements of a cone whose vertex is OO′, and base E*h*F) they are the vertical projections of the element of the conoid.

3°. *The front top edge*, BXF — B′C′, is assumed to be the intersection of the conoidal front-face of the wall with a plane, parallel to the straight directrix OF, and having a slope of 3 to 2. BF is then found by projecting down B′, *y′*, X′, etc., intersections of B′C′ with given elements, upon the horizontal projections of the same elements, as at B, *y*, X, etc.

The back of the wall, made of the given top thickness AB,

may then be made concentric with BF, as seen in plan, by making it tangent to any sufficient number of arcs, of radius AB, and with their centres on BXF.

The top edge of the back is here assumed to be the intersection of the *vertical cylindrical surface* of the back, with a plane perpendicular to **V**, and whose vertical trace is A′C′.

The top surface of the wall will thus naturally be a warped surface, having BF — B′F′ and AC — A′C′ for directrices, and **H** for its plane director.

The face-joints. — These shall be the elements just found, for the *heading* joints; while the *coursing* joints shall be equidistant horizontal sections, *m*F — *m*′T′, etc., which (115) are ellipses.

The joint surfaces. — With the quick curvatures arising from small dimensions and large batter, as in the present example, these surfaces should be normal to the face of the wall along the lines just fixed upon as the joints of the face.

The coursing surfaces, or beds, will thus be warped surfaces, which, for the joint *m*F — *m*′T′, for example, will be generated by a line *m*A — *m*′B″, normal to EB — E′B′, and moving upon *m*F — *m*′T′ as a directrix, so as to continue normal to the face of the wall. Now, since there can be but *one tangent plane at any one point* of a surface, and but one perpendicular to a plane at any one point, there can be but *one normal line to a surface at any* one point; hence, the warped surface thus generated is determinate.

The heading surfaces will be the normal hyperbolic paraboloids along the elements of the conoid (117).

5°. *Construction of the joint surfaces.* — If two lines are perpendicular to each other, and *one* of them be parallel to a plane of projection, their projections on that plane will be perpendicular to each other. Now the *elements*, as *lu* — *l*′*u*′, of the conoid, are parallel to **V**; hence, the *vertical* projections, *r*′*s*′, *n*′*q*′, etc., of the normals to the conoid, along *lu* — *l*′*u*′, will be perpendicular to *l*′*u*′. And the like is true for the other elements. Again, the normals at points of the ellipses *m*F, etc., are perpendicular to the tangents to those ellipses at the same points; hence, *rs*, *nq*, etc., *o*Q, etc., are perpendicular to the tangents at *r*, *n*, *o*, etc., respectively, to the ellipses H*r*F, *mn*F, etc.

Projecting A, Q, R, etc., upon the vertical projections,

$m'B''$, $o'Q'$, $k'R'$, etc., normals at m', o', k', etc., we have $B''Q'R' \ldots . T'$, tangent to $m'T'$ at C,T', for the vertical projection of a coursing joint on the *back* of the wall; and $mAFC - B''m'T'S'R'$ as the normal coursing surface containing the ellipse $mF - m'F'$.

Likewise, projecting p, q, s, etc., upon the perpendiculars to $l'u'$ at l', n', r', etc., we find $p'q't'$, a heading joint, upon the back of the wall.

6°. *The tangents* at o, n, r, etc., to the ellipses mF, etc., and to which the *normals* oQ, etc., are made perpendicular, may be drawn in various ways, as most convenient for each point.

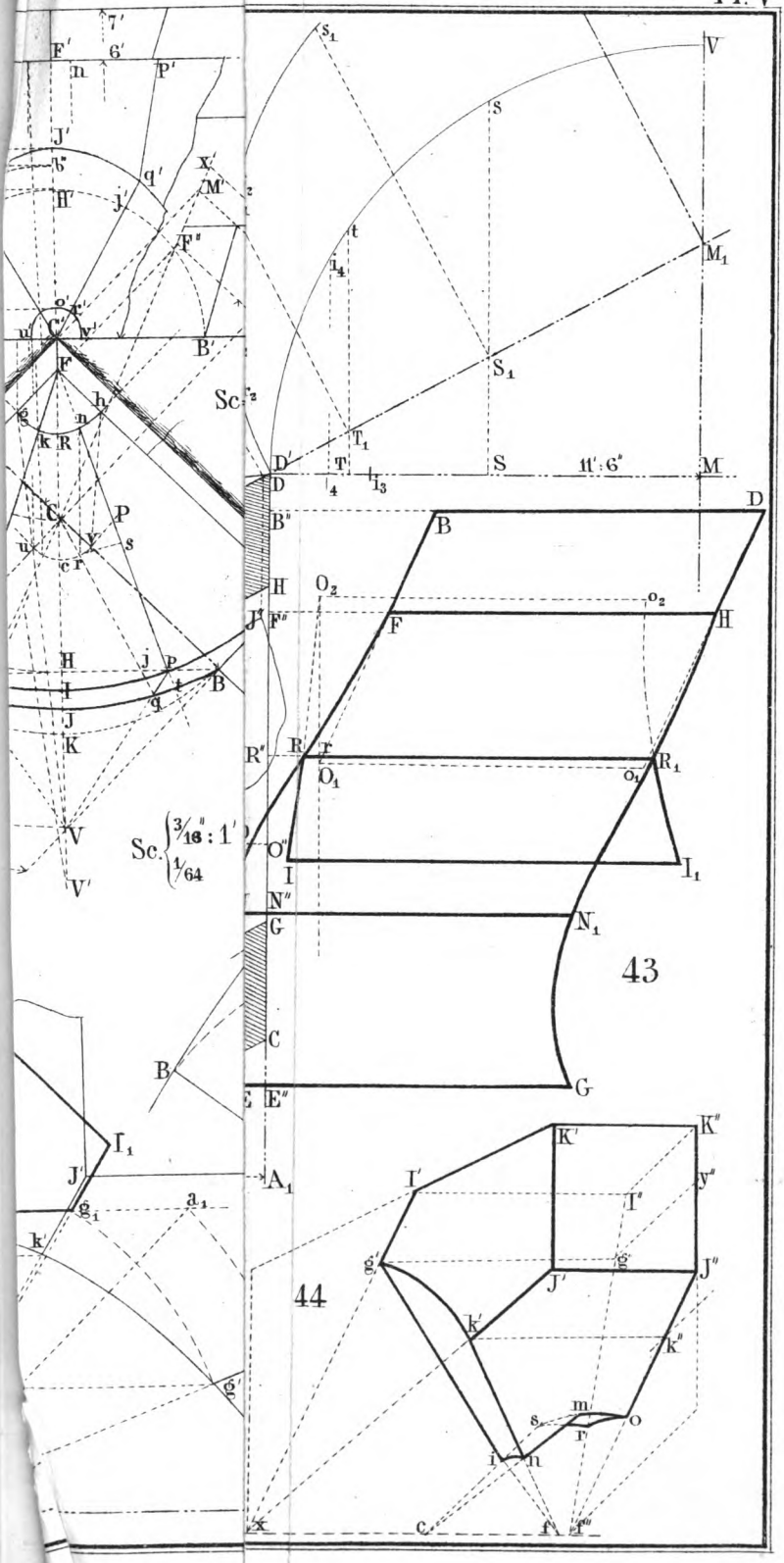
1*st*. That at zz', to the ellipse mF, is here drawn by the *method of bisecting* the angle MzN included by lines from z to the foci, one of which, f, is shown, and both of which are at the intersections of an arc with radius Om, and centre F, with the transverse axis mO; produced to find the other focus f_1.

2*d*. That at n, to the same ellipse, is drawn by the *method of revolution;* the quadrant, EF, being the projection of mF after a certain revolution about OF. Then n appears at l, and lG is the revolved position of the required tangent, which, by counter-revolution, appears at Gn.

3*d*. The tangent, Jr, is likewise found from Jl, where the revolution of the ellipse, HF, take place about HO, till it appears as a circle of radius OH.

4*th*. If adjacent figures were not in the way, so that other quadrants of each ellipse could be shown, we might proceed as follows, by the *method of conjugate diameters*. Thus, at o, for example, draw Oo and any *chord*, *parallel* to it, and the tangent at o to moF would then be parallel to the line from O to the middle point of this chord, for such line would be the diameter conjugate to Oo.

7°. *Approximate joint surfaces.* — With the considerably *larger* dimensions, and *less* declivity of face, which would be generally found in practice, the part from X to F would be nearly a vertical plane, and from BE to X, nearly a conical, or even an almost vertical cylindrical surface. Hence, in such a case the *coursing surfaces* could be safely *horizontal planes;* and the heading surfaces could be planes. That through the element $lu - l'u'$, for example, might be a plane, determined by the element together with the horizontal element, at K', of the top surface.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II. *The Directing Instruments* and their *Application.* As *every* surface, except the back, which would generally be left rough, of all the stones between UU′ and *gg′* is warped, no *patterns* can be used. It would therefore be best to *begin*, at least, by working some one surface by the method of squaring (105) from the sides of a circumscribing prism, so far finished as to admit of the application of this method.

Having the front, for example, thus wrought, *beds* could be made square with it by keeping one arm of the square (No. 2) on an element of the front, and the other on an element of the bed.

With the often admissible approximate *plane* joints, already described, the operations would be much easier, as patterns of all the faces except the conoidal front could be easily found and applied.

These general guiding observations, added to previous examples, will enable the student to construct whatever instruments may be necessary.

EXAMPLES. = 1°. Construct the front elevation of this wall, and an isometric, or oblique projection of one of its stones.

2°. Construct the wall when OE is less than OF.

STAIRS.

119. Stairs vary in form; first, *as a whole*, depending on the form of the space which they occupy; second, *in detail*, that is, in the form and arrangement of the separate steps.

Certain *practical conditions* and *geometrical principles* are, however, common to all cases; hence, a single example of a general case, fully explained, will serve as a standard, from which variations may be made to any particular forms.

120. General Geometrical Principles. — All stairs may be divided into two principal kinds.

1st. *Straight stairs;* in which the height, or *rise*, and the width, or *tread*, are each uniform, on each and all of the steps.

2d. *Winding stairs;* in which the rise remains uniform, while the tread is variable at different points of each step. In this sense, winding stairs which consist only of successive short flights of straight stairs running in different directions, are not included.

121. Winding stairs are, again, of *two species* : —

First, those which wind around a single vertical axis from which the edges of the steps radiate.

Second, those which radiate around no single central axis from which the steps radiate.

In the former, the tread is uniform on a line of ascent taken at *any* given distance from the axis. In the latter, there is but one such line, and it is taken at that distance from the hand-rail which any one would naturally choose in passing up or down the stairs, and may be called the *line of passage.*

Stairs mostly straight are often partly winding, at one or both ends, and will then be classed under one or the other of the varieties just indicated.

122. In winding stairs of the *first* species, the natural line of passage upon them is obviously a common or circular *helix*, as in Pl. VII., Fig. 50; where, if a horizontal and a vertical plane be passed through each element, they would evidently intersect each other so as to form the steps of such stairs as would wind around the axis of a cylindrical pit.

In winding stairs of the *second* species, the horizontal projection of the line of passage will not be a circle as in Pl. VII., Fig. 50, but some other curve, as LP, Fig. 8, better conformed to the ground area covered by the stairs.

123. Now let LP, Fig. 8, be the horizontal trace of a vertical cylinder, on which a point, m, moves so that the horizontal and vertical components of its motion are equal, that is, so that if $mm_1 = m_1m_2$, etc., the heights of m_1 above m; of m_2 above m_1, etc., will be equal. The point m will thus generate a *helix*, h, but of a more general kind than the circular helix, Pl. VII., Fig. 50.

124. Also if a *horizontal straight line* taken as a generatrix G, move upon this new helix LP, in the same manner as in Pl. VII., Fig. 50, the resulting surface will still be a helicoid, H, but of a more general form than the usual particular form shown in that figure.

But as the plan, LP, of h is no longer a circle, while the line G continues perpendicular to it as seen in plan, Fig. 8, the lines g, g_1, g_2, etc., horizontal projections of successive positions of G, will not pass through any one point, as at o in Pl. VII., Fig. 50, but will intersect each other, as in Fig. 8, so that when g, g_1, g_2, ... g_n, and thence their intersections, p, q, r, etc.,

become consecutive, $p, q, r, \ldots\ldots r_n$, will form a curve to which $g, g_1, g_2, g_n \ldots\ldots$ will be tangent.

125. We thus reach the following important conclusions; *foundation of the design* of stairs of every form.

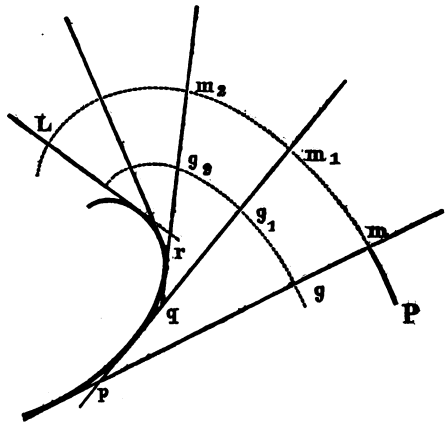


Fig. 8.

1°. The curve *pqr* is the horizontal projection of a *vertical cylinder*, C, which replaces the vertical straight line (axis) *o* — 0'12' in Pl. VII., Fig. 50.

2°. The curve LP is the horizontal projection of a general form of a helix, *h*, (123).

3°. The straight line *g*, then moves upon the helix *h*, and remains *horizontal* and *tangent to the cylinder* C. It thus generates a general form of helicoidal surface, H; such as forms the under surface of the stairs, and such as will contain the similar radial edges of all the steps.

PROBLEM XVI.

Winding stairs on an irregular ground plan.

I. *The Projections.* Let a landing at A*d*E, Pl. VIII., Fig. 60, provided for a door in the wall ES, be connected with a floor *three feet* higher at CH, by a flight of five steps, counting the upper level. And let these steps be built three inches into the wall, whose base is the segment HGFE of an irregular polygon.

Let the *line of passage* (121) be the arc CD of the ellipse, whose semi-axes, OA, and OB, are determined by convenience.

The tread of each step, measured from D on AC, is 12 ins. giving I, J, K, C. At these points draw normals to the ellipse, AC, as Dd, etc., which will be edges of steps. On these normals lay off 18 ins. inward from CD, to locate the inner end, cd, of the steps, or the circumference of the *well*, as it is called.

For firmer support, let each step extend as at Kg, or Jk (J_2k_1) and K_1g_1, 7 ins. under the next upper one; and then normals to CD at k and g, will be the under edges of the step JK, and, see k_2 and g_2, in the general under surface of the steps.

This under surface of the steps is a right helicoid, H, generated by Qkq moving upon a helix whose horizontal projection is CD, normal to it, and parallel to **H** as its plane director.

Normal joints. — These, perpendicular on Qq, etc., to the helicoid, H, should (by 118) be hyperbolic paraboloids. But it is a sufficient approximation to make them normal *planes* at some suitable mean point. They are here made normal at g, k, etc., points on the *helix* CD, for two principal reasons. 1st. Only the helix CD will be straight in development, it alone having equal arcs for equal ascents (123). Hence, normals at its points can be more easily drawn. 2d. All the curvatures are quicker within than without CD; hence the warped, and plane normal surfaces, will differ less if the normals be nearer cd than to the wall.

The development. — Proceeding as just indicated, make C_2D_2 = CD, and C_2C_1 = the total rise = 3 ft. and C_1D_1 will be the development of the helix over CD, and which cuts the front edges of the steps.

Next, after dividing C_1D_1 into four equal parts, make $J_1J_2 = \frac{1}{3} C_1C_2$; $J_1K_1 = JK$; $J_2k_1 = Jk$; $K_1g_1 = Kg$; and g_1g_2 and k_1k_2, each, for example equal to $\frac{1}{2} J_1J_2$, when g_2s_1 parallel to C_1D_1, will be the development of that helix, projected in CD, which is in the helicoid H.

Finally, at g_2, k_2, etc., draw g_2f_1, k_2h_1, etc., perpendicular to C_1D_1, and they will be lines of the required normal planes to H, at g, k, etc. Do the same for the other steps, and the figure $D_1s_1t_1J_1$ will be the complete development of the section of the steps made by the vertical cylinder whose base is CD.

Then, to complete the plan, make $gf = kh = g_1f_1$, etc., and the lines as M*hm*, parallel to Q*kq*, etc., will be the traces of the normal planes to H upon the treads.

II. *The directing instruments*, besides Nos. 1 and 2, are these:—

No. 3, the pattern, ML*ml*, of the top of a step.

No. 4, the pattern, M′L′R′, of the wall end of a step. This is found by projecting the outer end, QL, of a step upon a parallel plane, as shown; making all the vertical distances equal to the like ones on the development; $Q'Q_1 = g_1g_2$; $P'P_1 = K_1K_2$ (greater than J_1J_2, since K_1g_1 is greater than $k'k_1$).

No. 5, the pattern, $n_1d_1b_1$, of the inner end of a step. This, for the step E*d*rR, is the development of its inner end. Then r_1n_1; r_1l_1; r_1d_1, etc., equal *rn*, *rl*, *rd*, etc., on the plan; and the heights, l_1l_2, etc., equal those on the delopment K_1K_2, etc.

No. 6 is a normal joint bevel, giving the constant angle $J_2h_1k_2$.

III. *Application.* — Having chosen a sufficient block, bring its intended top to a plane, and mark its form by No. 3. Work the rise and the end, square with the top, using Nos. 4 and 5 to give the forms of the ends. Work the normal joints by No. 6, and the helicoidal under side by No. 1, applied on points transferred from the drawing, where elements would meet *ql* and QL, for the step IJ, for example.

Other Forms of Stairs.

126. *Other stairs* are, for want of space, merely suggested by the steps, illustrated in Figs. 61, 62, which are both adapted to *circular* stairs; that is, those placed in a cylindrical case. They are contrasted in the manner of support. In Fig. 61, which shows a plan, elevation of the back edges, and a development of the cylindrical outer end BD, the central open cylinder, or well, is filled by a *core*, composed of the cylindrical wings, or ears, O, solid with the inner end of each step. The core is sometimes larger, and then solid, and with the steps indented into it, as at the outer ends in Fig. 60.

Fig. 62 is an oblique projection of one step of circular stair with an open well; and the steps are supported by an ear CE

at the outer end whose whole thickness, F*e*, is indented into the wall, giving them a wide horizontal support.

EXAMPLES. — 1°. Observing that the curves, CDA and XY, Pl. VIII., Fig. 60, have the relation of involute and evolute to each other, represent, with the patterns, stairs in which XY shall be assumed.

2°. Stairs in which *cd* shall be assumed.

3°. Construct stairs whose steps shall be formed as indicated in Fig. 62.

4°. Construct circular stairs in a cylindrical case, with a central post formed of steps like that of Fig. 61.

5°. In a flight of five, or more, steps against one reach of wall, as GF, Fig. 60, construct the intersection of that wall with the helicoid, H, of the under surface of the stairs.

CLASS IV.

Structures containing Double-Curved Surfaces.

PROBLEM XVII.

A trumpet bracket with basin and niche.

I. *The Projections.* These are a plan, RrE ; a front elevation A′B′E″ ; and a sectional side elevation, O‴R″E⁗.

1°. *The outlines of the bracket.* Rrr_1 Pl. IX. Fig. 63, is a wall of the general thickness QK; but, where the bracket is attached, of the additional thickness, r_1s. The front of the bracket is composed of two cylindrical surfaces; one vertical, with the radius OA; the other horizontal, with its elements, AB, *na*, etc., parallel to the ground line A′B′.

The form of the latter cylinder is made to depend on the given curve, AEB — A′E′B′, of their intersection, where A′E′B′ is a semicircle. This curve would be the intersection of the vertical cylinder, with a cylinder of revolution whose vertical projection would be A′E′B′, as shown more clearly in the auxiliary Fig. 64, where AEB — A′E′B′ is the intersection of the vertical cylinder, CD — C′D′, with the cylinder, UT — A′E′B′, which is perpendicular to the vertical plane. By Theorem I., the projection, E‴O‴, of the intersection AEB — A′E′B′, on a plane, as O‴X, parallel to the plane, EF, of the axes of the cylinders of revolution, is a hyperbola. The vertical cylinder is then cut away to this hyperbolic profile, E‴O‴; so that the face of the bracket within A′E′B′ is a cylinder parallel to the ground line and with a hyperbolic right section, or base O′E′— O‴E‴.

The top of the bracket is a plane annular surface, between AEB — A″B″ and the circular edge, of radius OH, of the hemispherical basin, H′G′I′ — J′G″J″.

2°. *The joints of the bracket.* — Having found O‴E‴, as may be seen by inspection, divide A′E′B′ into equal parts, here five, and draw the radial joints, as h'O′; c'O′, etc., limited, to avoid thin edges, by a semi-cylindrical stone of radius UT — O′T′, = R″r'''. Then —

To find intermediate points in the joints on the vertical cylinder. — Assume b' as such a point. Its horizontal projection is b, and auxiliary projection b_1, which by revolution appears at b'', intersection of $b_2 b''$ and $b' b''$.

To find intermediate points in the joints on the horizontal cylinder. — Here it is the horizontal projection of the point that needs construction. Assuming d', for example, project it at d'', on $O'''E'$, thence to sX and revolve upon Xm; whence project upon $d'd$ at d. Or, by the *method of transference*, make $d_1 d = d''' d''$ (50).

By the same method, applied to b'', make $b''' b'' = ob$, to find b''. Also, make $h'' e''' = hh'''$, to find h''. In similar ways the points of all the joints of the bracket can be found; as at $e''' h_1'' = h''' h_1$, in finding h_1, a point of the basin joint, $h_1 gk$ — $h' g'$ — $h_1'' g'' h_1'''$.

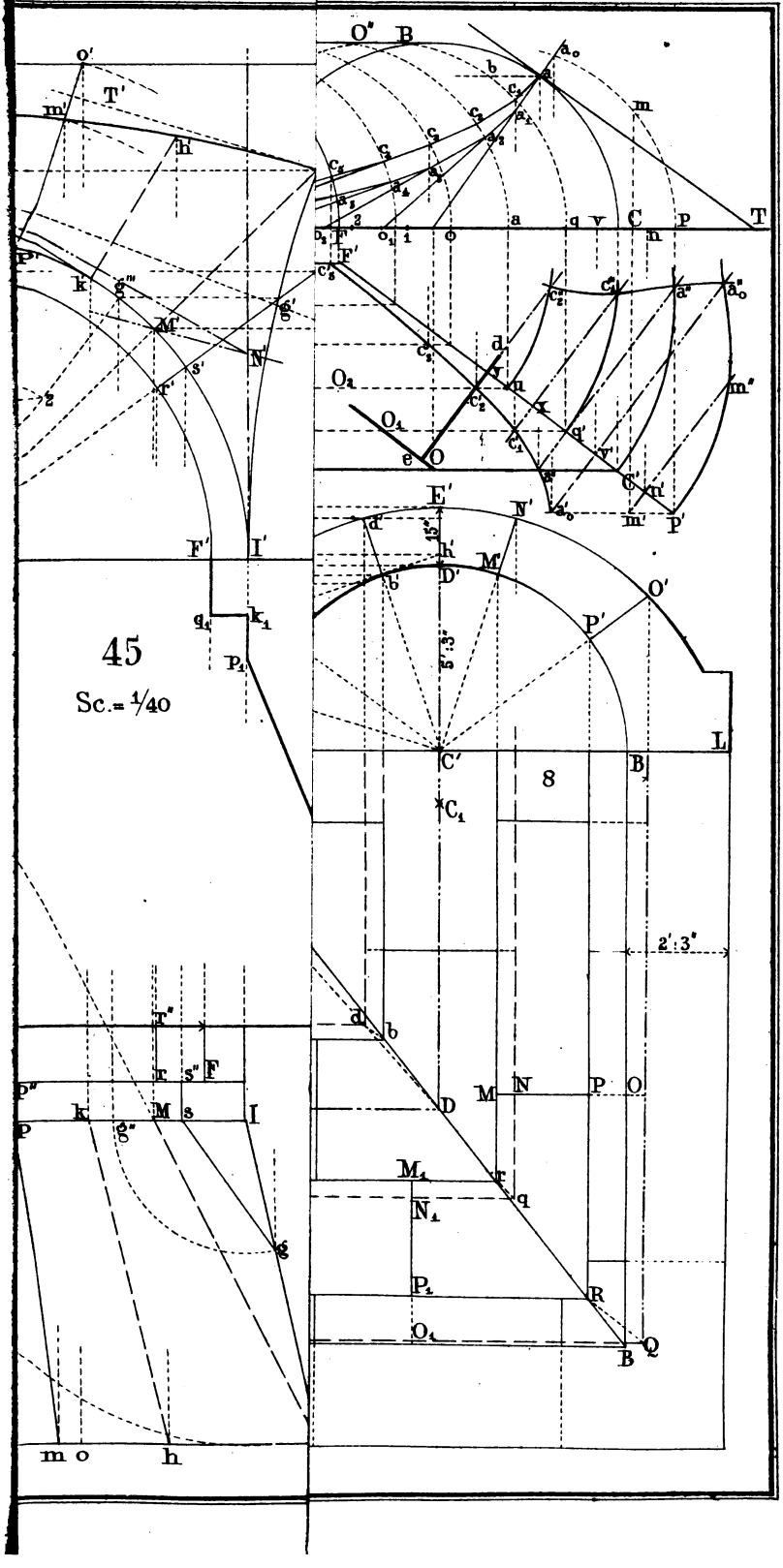
The horizontal projections, as $h_1 gk$, of the basin joints, above gg', on the vertical semicircle, H'G'I', of the basin, are found by horizontal circles, as that with radius $j' l'$, each of which will contain four points of the two basin joints, of which points $i_1 i'$ is one.

3°. *The Niche.* — The niche is a vertical semi-cylinder, CFD — C'C''D'D'', covered by the quarter sphere included between the horizontal semicircle, CFD — C''D'', and the vertical one, CD — C''E''D''. The joints of the spherical part are circular, and in planes which radiate from the diameter, OF — O'' — O''''F''; and are limited by the cylindrical stone, O''L'P''.

Intermediate points of these joints, as N', are readily found on the plan and side elevation, in various ways. Thus, MM_1 — M'N'M'$_1$ — M''N'' is a vertical circle through N', which point is thence projected upon MM_1 at N, and upon M''N'' at N''. This circle might have been made horizontal through N'; or the plane of the joint might have been revolved about OK — O''K' as an axis, when the circular joint would have fallen on K'E'' and N' at n''. Then make $n_1 N = n'' N'$, which will give N, as before.

Varied constructions are useful, in case some one of them does not conveniently apply to certain points. Points, as K', being projected at K, and K'', and L' at L, and L'', KNL and K''N''L'' are the other two projections of the circular joint K'L'.

II. *The Directing Instruments.* — These will consist of pat-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terns of the plane and developable faces of all the stones, with a sufficient number of bevels to insure the correct relative *positions* of all the surfaces.

After the numerous preceding examples of developing the convex surfaces of stones of irregular shape, the construction of the required patterns can be made from the following general directions.

Take the stone, $A''h'\ V'v't'$, of the bracket, and $L'K'p'$ of the niche, as being the most irregular ones. Let each extend to the vertical plane back, Rr — $R''Q''$, of the wall. Develop the entire convex surface, both lateral and end faces, of each, as in previous examples. The following, including the necessary bevels, will thus be found.

No. 1, the straight edge.

No. 2, the square. Then for the bracket stone alone —

No. 3, the pattern of the back, $= A''h'V'v't'$.

No. 4, that of the top, $= A'''h_2h_3Hh_1hA$.

No. 5, that of the radial joint, $hiVuh_2h_3gh_1 - V'h'$.

No. 6, that of the opposite radial joint, $AA'''v_1vn$.

No. 7, that of the surface, Vvv_1u.

No. 8, the development of the vertical cylindrical surface, $Ahi - A''h'i't'$.

No. 9, the development of the portion, $nivV - n'i''v'V' - i''f''$, of the horizontal hyperbolic cylinder of the front.

Nos. 10 and 11, as shown on the figure.

The corresponding guides for the niche stone can readily be found.

To avoid the too acute angle at h_1 in the bracket stone, $H'G'I'$ might have been an arc of 120° or less; or the portion of the bracket above E' might have been a single stone thick enough to contain the basin.

III. *The Application.* This is, in the main, sufficiently obvious, from the description of the patterns, and the previous essentially similar cases. The back of both the bracket and the niche stone may properly be wrought first, since all the lateral faces are perpendicular to it. The order and manner of using the remaining guides may be left to the workman.

EXAMPLES. — 1°. Construct the niche alone.

2°. Construct the bracket alone, and without the basin.

THEOREM III.

The conic section whose principal vertex and point of contact with a known tangent are given, will be a parabola, ellipse, or hyperbola; according as the given vertex bisects the subtangent, or makes its greater segment without, or within, the curve.

In both the ellipse and the hyperbola, referred to their centres and axes, the subtangent is a fourth proportional to the abscissa of contact, and those segments of the transverse axis which meet on the ordinate of contact. That is, in Figs. 9 and 10,

$$CO : Oa :: OA : OT;$$

whence, by division,

$$CO : Oa - CO :: OA : OT - OA;$$

or

$$CO : CA :: OA : TA$$

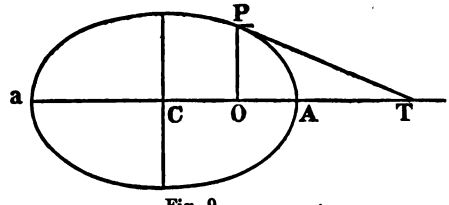


Fig. 9.

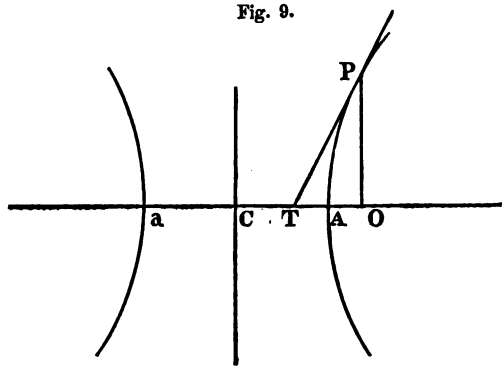


Fig. 10.

Now, always, in the ellipse, $CA > CO$

$\therefore$ $TA > OA$

But in the hyperbola, $CA < CO$

$\therefore$ $TA < OA.$

In the parabola, $AO = AT.$

PROBLEM XVII.

The hooded portal.

The Projections. — Pl. IX., Fig. 65. This construction is known in France as the recessed gate of St. Anthony, it being that gate near to the Bastile, which led to the suburb called St. Anthony. The figure represents a broken *plan* and *front elevation*, and a vertical section through the axis of the portal. Its design is to give to a portal, closed by rectangular gates, something of the grander effect of a semicircular topped portal, closed by gates of like form, as in Pl. VI., Fig. 45.

Let A*c*F*m*VO be half of a horizontal section of the portal, below its top, V′F′. Then, as in Problem XI., the entire passage embraces three parts, of which one half of each is as follows: the portal proper, EV*m*F; the gate recess, E*c*′′*ce*; and the converging embrasure, O*ec*A, flanked by the jambs, one of which is A*c*.

The tops of the two former parts are horizontal planes. The latter is covered by the very peculiar *double-curved surface*, characteristic of the structure, and generated by a *variable semi-ellipse*; which, starting from the semicircle of radius OA, as its initial position, moves so that its centre shall remain on O*e* — O′, one vertex upon A*c*, and another upon a fixed ellipse, C′′*p*′′*e*′′, whose axes are O — O′C and O*e* — O′. Hence, this elliptical generatrix varies from the semicircle of radius OA to the straight line 2*ec*. Thus, when this ellipse has reached the plane *ap*, its semi-axes will be *ap* and O′*p*′, = *qp*′′. Likewise, *bf* and O′*f*′ are the semi-axes of another position of the movable ellipse.

We may note in passing that, beginning at the vertical plane OA, the horizontal axis will diminish more rapidly than the vertical one, until we reach the point of contact of that tangent to *c*′′*p*′′*e*′′, which makes the same angle with O′′D′′ that A*c* does with OA. Hence, as O′A′C′ is a circle, a few positions of the movable ellipse near it will have their longer semi-axes vertical.

The joints, as L′J, in the vertical plane portion exterior to the recess, are straight, and radial to the semicircle of radius O′A′.

The face joints, as L′M′, within the recess, are best made

normal to both of the limiting positions of the variable elliptic generatrix. This result may be obtained by making these joints, as seen in front elevation, as circular arcs, tangent, as at L′, to the radial joints, L′J, etc., just described, and with their centres on O′A′. But such joints will divide 2O′*c*′ unequally. If the latter result be thought undesirable, 2O′*c*′, the top line of the gate recess, may be divided equally, as in the figure, and the joints may be either *conic sections* or *curves of two centres*, tangent, as at L′, to the radial joints, and, as at M′, to vertical lines.

The construction is illustrated in the joint K′G′, which (Theor. III.) is elliptical, since the greater segment, K′O′, of the subtangent, O′*o*, is exterior to the curve. Bisecting the chord G′K′, and joining its middle point, *n*′, with *d*′, the intersection of the tangents at G′ and K′, we have, by a property of the ellipse, *d*′*n*′ as a diameter; which therefore meets the axis, O′A′ (and the opposite symmetrical diameter), at X, the centre of the curve; whence the shorter axis can be found from the property of the subnormal, *o*S, expressed by the proportion,

$$X\!o : o\mathrm{S} :: a^2 : b^2,$$

where *a* is the semi-transverse axis, XK′; and *b*, which equals the semi-conjugate axis, = AB, Fig. 66, is found by the usual construction from elementary geometry.

Examining the joints M′L′ and P′Q′, we find M′O′ > M′*h*′, but P′O′ < P′*r*′. Hence, by Theor. III., the joint M′L′ should be elliptical, and P′Q′ hyperbolic; but the differences, M′O′ — M′*h*′ and P′O′ — P′*r*′, are so small, that they are here made with sufficient accuracy as circular arcs, whose centres are on O′X.

The horizontal projections of the joints are found by projecting down their intersections with the contours of the surface, made by the vertical planes, *pa* and *fh*, as is fully shown for the joint K′I′H′G′; whose horizontal projection is KIHG.

II. *The Directing Instruments.* — Most of these can be sufficiently indicated by a description of the most irregular stone of the structure; that whose vertical projection is R*y*′F′K′G′, and which is more clearly exhibited in the oblique projection, Fig. 67, like points having like letters in both figures.

The many surfaces of this stone are: —

1°. The vertical rectangular plane side, YyUY′.

2°. The vertical plane back, Y′Uk.

3°. The vertical plane front, GRYyZ.

4°. The horizontal plane base, UkK″$c'$$zZy$.

5°, 6°. The horizontal plane top, RYR′Y′; and small horizontal plane surface, Kcc'K′.

7°, 8°, 9°, 10°. The four minor vertical plane faces, K′K″k, K′K″$c'c'$, czc'', and AZz, respectively, in the portal, gate recess, and jamb.

11°. The oblique plane surface, GRG′R′.

12°. The elliptic cylindrical surface, G′k'KHG. Fig. 68, is the development of the like surface on M′L′, joined with the plane portion on L′J.

13°. The double-curved surface, AcKHG.

The last surface being non-developable, no pattern of it can be made, but templets fitted to any of its vertical *sections*, parallel to C″e'', or to $f'b'$, or to its horizontal sections, can be made.

These templets, with patterns, easily made, of the other surfaces, and the square and straight edge, will be ample guides in working this stone.

III. *Application.* — First form the surface, YY′Uy, it being the largest and simplest; next the back; and then the base and front, and all the other plane surfaces, each of which is square with one or more of the others.

The cylindrical surface, GG′Kk', may then be wrought square with the back upon G′k', as a given edge, or directrix, previously found by the pattern of the back. Or it may be wrought by templets fitted to the profile, R′G′k'. Its edges may then be scored on the stone by a pattern corresponding to that of the cylindrical joint on M′L′, shown in Fig. 68.

These operations will give all the bounding edges of the one remaining surface, which is double-curved. After approximately hewing out this portion of each of the stones, they can be accurately put in place, since *all* the other surfaces of each will have been previously completed. The total double-curved surface of the recess can then be wrought at once, by means of the templets, last described in the list of guiding instruments.

EXAMPLES. — 1°. Construct the figure with two centred joints in the front elevation.

2°. Make an isometrical or an oblique projection showing the under side of the stone shown in Fig. 67.

3°. Make like projections of the stone M'JTR.

PROBLEM XIX.

An oblique lunette in a spherical dome.

I. *The Projections.* — A *lunette* is formed by the intersection of two arched spaces, both of stone, and of unequal heights, so that the groin curves will be of double curvature.

1°. *Arrangement of projections.* — Pl. X., Fig. 69. These are a plan, and two elevations, on two vertical planes, V and V_1, at right angles to each other, and whose ground lines are respectively O'X and O''X. As in all similar cases, the projections of any point on V and on V_1, will then be at equal heights above O'X and O''X.

Given parts and dimensions. — In the *plan*, the circles, OA of 11 ft. radius, and OH of 13' : 6'' radius, are the horizontal traces of the interior and exterior surfaces of the dome. The former is a hemisphere; the latter, partly cylindrical, as indicated in the section shown on the plane O''X, is there generated by H'H'', 6' : 10'' high. The radius, D''x, of the extrados, is 16 ft., where x is 3 ft. below the centre, O'', of the intrados. The elevation on O'X shows a right section of the arch, its inner radius 4 ft., its outer one 8 ft., its thickness at the crown 1' : 6''; and the perpendicular distance of its axis, o_1o', from the diameter, HO, 6' : 3''.

From these data all the remaining constructions are made.

2°. *The groin.* — Any horizontal plane will cut a horizontal circle from the sphere, and two elements from the arch, which will meet that circle in two points of the groin. Thus, the plane, $a'_1m(m_1v)$, cuts from the sphere the circle of radius Oa_1 ($=vy$) and from the arch the two elements, of which one at a'_1, being projected on H, intersects circle Oa_1 at a_1, as shown, and thence gives its side elevation a_1 on vy. Other points being found in the same manner, give the groin curve aca_5 — $a'c'a'_5$ — $a''c''a''_5$.

3°. *The horizontal projection of the groin is an arc of a parabola.* — To prove this, refer the intrados of the sphere and cylinder to the three rectangular coördinate axes: OH_1, as the axis of X; OH, as the axis of Y; and the vertical at O, as

the axis of Z. Then, neglecting the usual negative sign of ordinates to the left of the origin, O, as not relating to the *form* of the line sought, we have for the point a_1a_1', for example,

$$(Oh)^2+(ha_1)^2+(h'a_1')^2=R^2$$

where R = the radius, OA, of the sphere.

That is $$x^2+y^2+z^2=R^2. \qquad (1)$$

And as the like is true for every point of the sphere, (1) is called the equation of the sphere, referred to its centre.

Again, $$(o'h')^2+(h'a_1')^2=(o'a_1')^2=r^2.$$

That is, calling $O'o'=a$,

$$(a-x)^2+z^2=r^2. \qquad (2)$$

and as the like is true for every point of the cylinder, this is called the equation of the cylinder for the given axes of reference.

Now that points, as a_1a_1', may be common to both surfaces, and hence be points of their intersection, the x, y, and z of (1) and (2) must be the same. That is, (1) and (2) will both be true at once for the same point, so that we can substitute any term in one for the like term in the other.

Then, from (1), $$y^2=R^2-(x^2+z^2)$$

and from (2), $$(x+z^2)=(r^2+2ax-a^2)$$

whence, $$y^2=-2ax+(R^2-r^2+a) \qquad (3)$$

which, since z is eliminated, is the equation of the curve aa_5, in the plane XY. Also, the term in the parenthesis is constant, being made up of constants, and as a is a part of it, it may be written $2na$, and (3) then becomes

$$y^2=-2ax+2na=-2a(x-n) \qquad (4).$$

If now we shift the origin O to the left, on the axis of X, so as to make $$x=x+n,$$

(4) will become, $$y^2=-2ax \qquad (5).$$

Restoring now the neglected sign of x, we finally have

$$y^2=2ax \qquad (6)$$

the usual form of the equation of a parabola lying, as a_5a does, to the *right* of its vertex taken as the origin. The curve aa_5 is, therefore the arc of a parabola, of which HO is the axis.

4°. *Joint-lines and surfaces of the sphere.* — The coursing joints on that part of the sphere which is independent of the lunette, are horizontal circles, IQ — I″Q″, GR — G″R″,

etc., found by dividing the meridian of radius $O''A''$ into an odd number of equal parts, — here eleven. The broken joints, $RQ - R''Q''$, etc., are arcs of meridians.

The *beds* of the dome voussoirs are the conical surfaces, as $P''Q''I''J''$, having the centre, OO'', of the sphere for a common vertex, and intersecting the spherical surfaces in the horizontal circles, as $I''Q''$ and $J''P''$.

5°. *Radial joint-surfaces of the lunette.* — These are wholly plane, and their edges are the intersections of these planes with the several surfaces of the dome and arch.

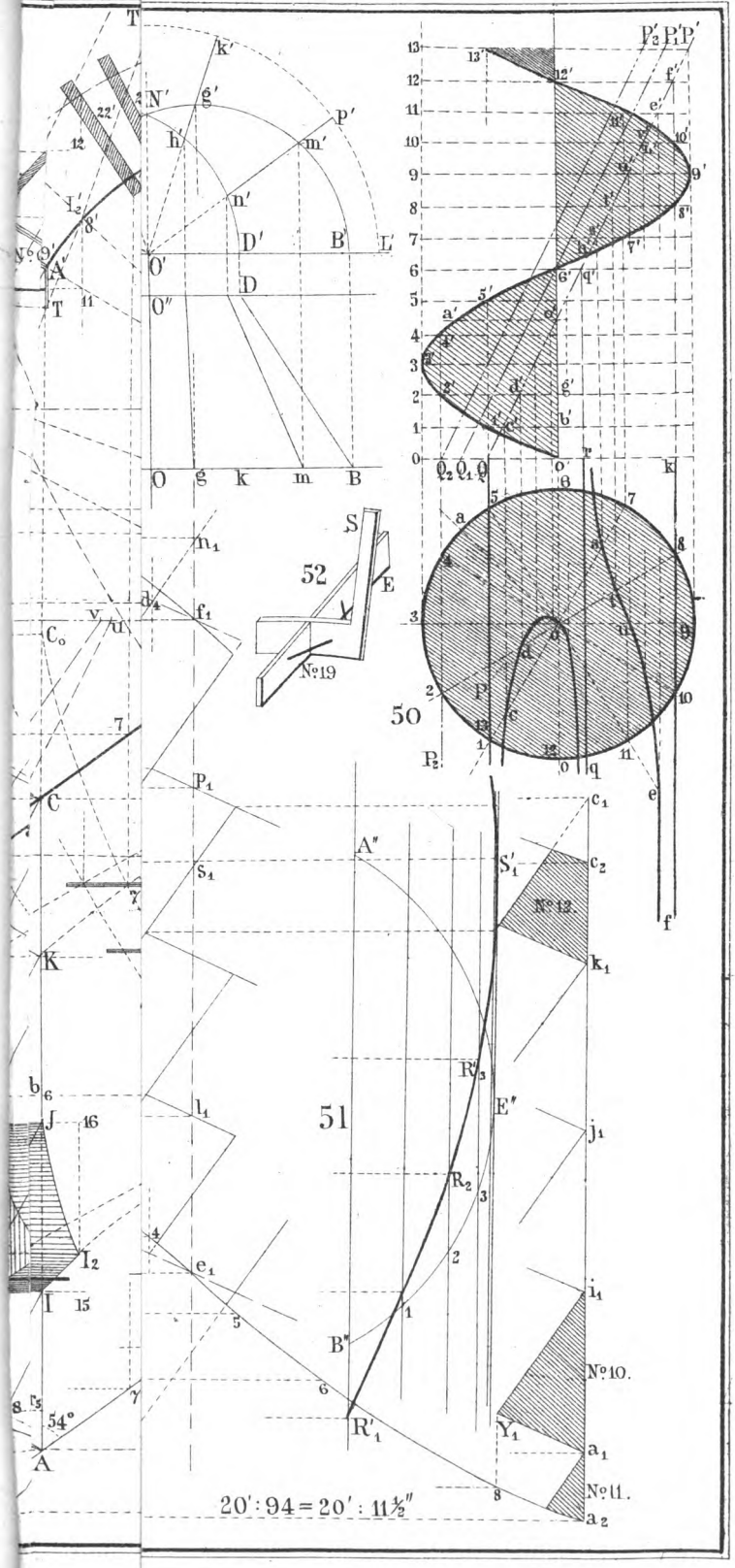
Divide the arc $a'c'a'_5$ so that a'_1 shall be lower than E'', the corresponding first one from A'', of the eleven equal divisions, on the dome section; here, into five equal parts. The reason for this will soon appear.

The lunette joints in the spherical intrados. — lb'_2 is the trace on **V** of the plane, $R''l_1$, of the horizontal circle $GR - G''R''$. This plane cuts the plane of the joint $o'd_2$ in a horizontal line at b'_2, which, by projection, gives b_2, and thence b''_2. Hence, $a_2b_2 - a'_2b'_2 - a''_2b''_2$ is one of these joints, showing that a_1 must be lower than E'', in order that there should be such a joint. The others are found in the same way.

6°. *The lunette joints in the conical beds of the dome.* — One of these is the intersection of the plane $o'd'_2$, with the conical bed, $C''G''R''R'''$. To find it, draw qd'_2, at the height of $C''q_1$, to give qd'_2, the trace of the horizontal plane of $Cd_2 - C''q_1$ upon **V**. As before, this plane cuts from the plane $o'd'_2$ a perpendicular to **V** at d'_2, which, in horizontal projection, gives d_2, on the horizontal projection, Cd_2 (C being projected from C''), of the circle considered; and thence d''_2. The joint sought is evidently a *hyperbola*, it being the intersection of the plane $o_1o'd'_2$ with the cone whose axis is the vertical at O, and whose slant is that of $G''O''$. Hence, make $p'O'o' = G''O''A''$, and $Op - O'p'$ is that element whose intersection with the plane, $o_1o'd'_2$, is the vertex, $p'p$, of this hyperbola; whose horizontal projection pb_2d_2 can now be more accurately drawn than without the aid of the vertex p. Finally, make $O''p'' =$ the height of p', and $p''b''_2d''_2$ is the vertical projection of the same hyperbolic joint, of which only $b_2d^2 - b'_2d'_2 - b''_2d''_2$ is real.

Any other hyperbolic joints are found in the same way.

Other lunette lines. — These are, for the same joint plane $o'd'_2$, the circular arc $d_2e - d'_2e' - d''_2e''$, on the spherical extrados.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Q₂ Q₁ Q
K
50
P₂
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q
e
f

$ef_2 - e'$, on the horizontal ledge, generated by D″H″; $f_2g_2 - e'g'_2$ on the cylindrical back of the dome; $g_2u_1 - g'_2$, on the extrados of the arch; $u_1u - g_2a'_2$, a radial edge in the arch; and $ua_2 - a'_2$, on the intrados of the arch.

The large diameter of the arch, as compared with the radius OA_1, carries the point $tt't''$ nearly out of the quadrant, OAA_1, unless, as shown at $t't''$, it be taken lower than the corresponding point bb'_1.

II. *The directing Instruments.*— These, besides Nos. 1 and 2, are patterns of all the plane, cylindrical, and conical surfaces of voussoirs, with certain bevels, as follows, taking for illustrations the stone between $o'd'_2$ and $o'g'_1$ of the lunette, and the stone R″Q″S″T″, of the dome.

No. 3 shows the real form, $a'_1g'_1a'_2g'_2$, of the *plane end* of the lunette stone, which is in the plane **V**.

Nos. 4, 5, and 6, Fig. 70, are patterns of the intrados (No. 5) of the same stone, and of the two radial plane joints when folded into the paper. Their construction is obvious, since like points have like letters with Fig. 69, and are found by ordinates from the vertical plane end, No. 3, in the plane **V**.

Useful bevels (not shown), would be No. 7, giving the angle $k''B''E''$; and Nos. 8 and 9, giving the positions of the plane beds on $a'_2d'_2$ and $a'_1g'_1$, relative to the intrados $a'_1a'_2$.

No. 10, the pattern of that plane end, MNW, of this stone, which is in the dome, is G″C″D″H″k''B″E″.

From MN to a_2b_2 is a spherical zone.

From N5 to b_1d_1 is a conical zone, No. 11.

From 5W in the plane B″k'' to d_1f_1, is a horizontal plane surface, No. 12.

From the vertical line, W — k''H″, extends the vertical cylindrical back, No. 13, of the dome, intersected by those surfaces of the lunette which are parallel to its axis $o'o_1$.

The three remaining surfaces of the part of the stone in the dome, are the plane annular portion, No. 14, generated by D″H″; the spherical portion generated by D″C″, and a conical portion, No. 15, generated by C″G″; all starting from the plane OW, and all limited at their intersections with that portion of the stone which is in the arch.

Patterns of these surfaces, so far as developable, may readily

be made; also bevels, conveniently giving the position of their horizontal edges, relative to the *end* in the plane OW.

Thus this very irregular stone has *thirteen faces*, plane, cylindrical, conical, and spherical.

To gain as full an idea of it as drawings alone can give, complete its projection on V; and make two or more isometrical, or oblique projections of it.

For the proposed stone of the *dome*, the pattern, No. 16, C″G″I″J″, of its vertical plane end will be needed; and those of its conical beds, as P″Q″V″S″, Nos. 17 and 18.

No. 17, for example, Fig. 71, is the development of R″R‴U″T″, found by describing the arcs from O, with radii equal to O″G″ and O″C″, and by making R″T″ = RT from the plan.

Finally, bevels like No. 19, will be useful, giving the relative positions of *elements* of the conical beds, and great circles of either the intrados or extrados of the dome. And a templet, No. 20, should be cut to an arc of a great circle of the spherical intrados.

III. *Application.* — For a stone as irregular as that of the lunette, the method by squaring (105) is preferable, if not indispensable. Then form a right prism, the pattern of whose base shall be the horizontal projection, $b_2a_2uxg_1$WM, of this stone; and upon whose rear and lateral faces the two plane heads can be marked by Nos. 3 and 16.

Next, the intrados and plane joints of the arch portion of the stone can readily be made square with the back by No. 2, and formed by Nos. 4, 5, and 6.

The plane, W$f_1d_1$5, is readily made; square with the end on MW, and marked by No. 12; the cylindrical back, square with the last surface, and marked by No. 13; and the spherical surface, MN$a_1a_2b_2$, by Nos. 19 and 20.

Pendentives.

127. In connection with domes, the related subject of square areas, covered by spherical surfaces, may be noticed; though detailed figures must be omitted for want of room. Pl. X., Fig. 72, shows a skeleton sketch of such a design, which is sometimes adopted on account of the stately appearance of a dome-like ceiling. Here let ABCD be the half of a square floor, of

which the circumscribed circle, of radius OB, is the base of a hemisphere. The four walls of the room will then be bounded by vertical small semicircles, as BC — A′Q′D′; the ceiling F′H′E′, within the circle of radius OA will be a spherical segment; and the four areas like ABI will be covered by spherical gores, shown more clearly at A″B″I″, in the elevation made on a vertical plane, whose ground line is *mq*, perpendicular to the diagonal BO.

128. *Two joint-systems.*—The joints of the spherical surface may then be either (*a*) *horizontal small circles*, and *vertical meridians*; or (*b*) *vertical small circles*, and meridians, all having BO — B″ for a common diameter; the beds bounded by the small circle joints being *conical* in both cases.

EXAMPLES.—1°. Make figure 69, on a scale of $\frac{1}{20}$, or larger, and with the arch smaller in proportion.

2°. The same with the two elevations side by side.

3°. The same, with the axis of the arch coinciding with a horizontal diameter of the dome.

4°. Complete the projection of the dome on V.

5. Construct the dome with pendentives, Fig. 72, in detail on a large scale, and by each joint-system.

SPIRALS.

129. A few observations on the spirals found in the next problem are here added, as they may not be conveniently accessible elsewhere.

A SPIRAL is a *plane curve*, generated by a point which has two simultaneous motions, or, more precisely, whose actual motion can be revolved into two components; one, a rotary motion, around a central point called the *pole*; the other, a radial motion, outward from the pole.

130. **Illustrations. The spiral of Archimedes.** — This is the simplest of all the spirals; since each of the component motions

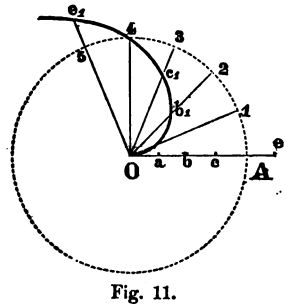


Fig. 11.

of the generatrix is uniform. Thus in Fig. 11, let O be the *pole*, and OA, the *initial line*, so-called, on which the successive equal increments of the radial movement are laid off. Then divide *any* circle, having O for its centre, into equal parts, as at 1, 2, 3, etc., and make $Ob_1 = Ob$; $Oc_1 = Oc$; $Od_1 = OA$, etc., and $Oa_1b_1c_1$. . . . . tangent to OA at O, will be a *spiral* of *Archimedes*. The distance of any point of the curve from the pole is called its *radius vector*.

In this example the circle is divided in 16 equal parts; hence, a circle of a radius, which we will call OA_1, comprising 16 of the parts of OA, from O, would be divided into the *same* number of parts as OA_1. As OA and any fractional part of the circle of radius OA may be divided into the same number of equal parts there may be an infinite variety of spirals of Archimedes.

Let the $\frac{m}{n}$th part of circle OA be divided into the same number of parts as the radius OA. Then, calling the radius vector $= r$; $OA = a$, and the arc, as A3, corresponding to any radius vector, as $Oc, = \theta$, we have, directly from the definition,

$$r : a : : \theta : \frac{m}{n}.2\pi a;$$

whence

$$r = \frac{a\theta}{\frac{m}{n}.2\pi a} = \frac{\theta}{\frac{m}{n}.2\pi} \qquad (1)$$

which is the general equation of the spiral in the form most convenient for use in drawing tangents to it by the *method of resultants*.

131. *Tangent to the spiral of Archimedes.* Differentiating (1) $\frac{d\theta}{dr} = \frac{m}{n} 2\pi$; where $\frac{d\theta}{dr}$, or $\frac{m2\pi}{n}$ is the *ratio* of the *rotary* and the *radial* components of the motion of the generatrix, the former being referred to the circumference of the circle whose radius is a. The application will be better understood by an example.

As we may generally make $m = 1$, write at once $\frac{d\theta}{dr} = \frac{2\pi}{n}$. Then let it be required to construct the tangent at b_1, in Fig. 12. Here, $n = 4$, since $\frac{1}{4}$ of the circle OA is divided into the same number of parts as are found on the line OA. Then lay off on Pb_1 produced, $b_1n = 4$, on any convenient scale; and $2p = 2\pi$, by the same scale, on the tangent at 2 to the circle OA; reduce the rotary component as estimated with the radius P2,

to its actual value, b_1s, parallel to $2p$, at b_1, by drawing the radius Pp. Then, b_1t, the diagonal of the parallelogram on the components, b_1n and b_1s, is the required tangent at b_1.

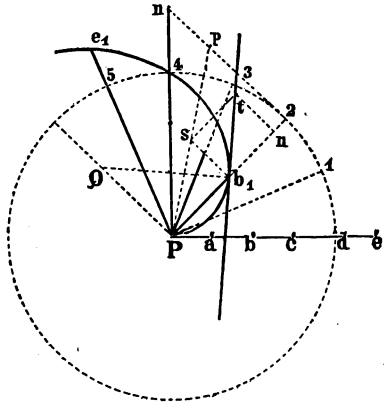


Fig. 12.

If $n = 1$, Pd, and the circle of radius Pd will be divided into the same number of equal parts, $\frac{d\theta}{dr} = \frac{2\pi}{1}$, and $2p$ would be 4 times $2p$, or b_1n would have been called 1 instead of 4.

132. *The subnormal method.* — Draw PQ perpendicular to Pb_1, and limited at Q by the normal b_1Q. Then PQ is the *subnormal*. Now the triangles PQb_1 and b_1nt are similar, and give,

$$\mathrm{PQ} : b_1n :: \mathrm{P}b_1 : b_1s \ (= nt)$$

whence

$$\mathrm{PQ} = b_1n \times \frac{\mathrm{P}b_1}{b_1s}$$

But if b_1n is made constant for each point, $2p$ will be so also; and hence, as we see from the figure, b_1s will vary as Pb_1; that is, the ratio $\frac{\mathrm{P}b_1}{b_1s}$ will be constant. Thus PQ is *constant*. Hence, having any one tangent, any other can be found as follows. Take, for example, the point e_1. Draw a perpendicular, PQ_1, not shown, to its radius vector Pe_1, at P, and equal to PQ. Then Q_1e_1 will be the *normal* at e_1, where the *tangent* will then be perpendicular to Q_1e_1.

133. The tangentoid spiral. — This is one of a series of curves known as the trigonometrical spirals, in each of which the radius vector, r, is some trigonometrical function of the angle, θ, between it and the fixed initial line.

The tangentoid spiral is that in which the increments of the radius vectors are *equal*, or *proportional*, to the increments of the *vectorial angle*, θ.

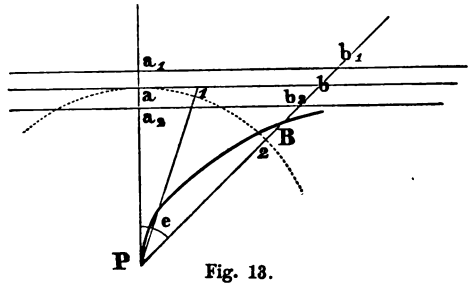


Fig. 13.

Thus, let P, Fig. 13, be the *pole*, and $Pa = a$, the *initial line*. Divide the circle P*a* at pleasure as at 1, 2, etc., then on P*b*, for example, make $r = PB = ab = Pa \times$ the tangent of BP*a*, and B will be a point of a tangentoid spiral, whose equation, simply expressing the construction, is, $r = a.\ \tan\theta$; or, if $a = 1$, then $r = \tan\theta$; that is, r is equal to the tangent of θ. But if, having drawn either a_1b_1, or a_2b_2 parallel to ab, we should make PB_1 (not shown), $= a_1b_1$; or, $PB_2 = a_2b_2$, we should find new forms of the spiral; where if $Pa_1 = a_1$ and $Pa_2 = a_2$, their equations would be $r = a_1.\ \tan\theta$; and $r = a_2.\ \tan\theta$, and r would be *proportional* to $\tan\theta$.

134. *Initial line a secant.* — In Fig. 13, the initial line, P*a*, is evidently tangent to the spiral at the pole, P. This is not always so.

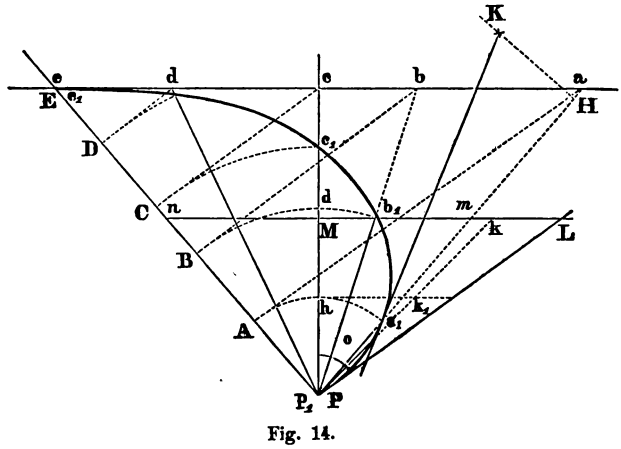


Fig. 14.

Thus, Fig. 14, let Ea and EP include any angle whatever, and let them be divided proportionally by parallels as Aa, Bb etc., including PL. Then arcs from A, B, etc., will meet the corresponding radials, Pa, Pb, etc., in points a_1,b_1, etc., of a curve which will still be a *tangentoid spiral.* For the increments, CD, CB, etc., of the radius vector, r, estimated from the circle Cc_1, where Pc_1c is perpendicular to Ea, are proportional to the increments, cd, cb, etc., of the tangents of θ, where θ is estimated from Pc.

Evidently P is the *pole*, and PL the tangent at P.

Then, in order to write $r = \tan\theta$; make ML = Pc_1; and take θ = MPL. Then ML = $\tan\theta$; and MP = $r = 1$.

Also the tangent increment Mm = the radial increment c_1h = CA, where Pm is the radius vector drawn through the point a_1. Likewise, Mn = CE, etc.

135. *The tangent line.* — From $r = \tan\theta$, $\frac{dr}{d\theta} = \sec^2\theta = \frac{\sec^2\theta}{1^2}$.

Now at a_1, for example, Pm = sec θ, and PM = 1. Then a third proportional (x) to PM, and Pm, will be the longer side of a rectangle; equal to (Pm)^2, and whose other side equals PM. That is PM$\times x$ = (Pm)^2. But the figures PM$\times x$ and PM^2 (= 1^2) having the common altitude PM, are to each other as x and PM. That is, $\frac{dr}{d\theta} = \frac{\sec^2\theta}{1^2} = \frac{(Pm^2)}{(PM^2)} = \frac{PM\times x}{PM^2} = \frac{x}{1}$.

Hence, make a_1H = the 3d proportional, x, for the *radial component* of the motion of the generatrix; and Mk = MP = 1 for the rotatory component, referred to the line on which $\tan\theta$ is estimated, and which will be reduced by the line Pk to its value, hk_1, for the circle of radius Pa_1. Then, making HK = hk_1, Ka_1 is the required tangent at a_1.

We now close with the following, which, besides its interest as a structure, embraces, as appropriate for a final problem, representatives of all the four classes of surfaces which form the main divisions of descriptive geometry, and of its application in this volume.

PROBLEM XX.

The annular and radiant groined arch.

136. *Supposed conditions.* — Suppose that a building for a private library or cabinet, or, if of suitable dimensions, for a locomotive engine house, is to consist of a gallery, annular in plan, and inclosing a central circular area. Also let the gallery be divided into seven compartments by arches, whose straight elements shall radiate from a vertical line at the centre of the central area.

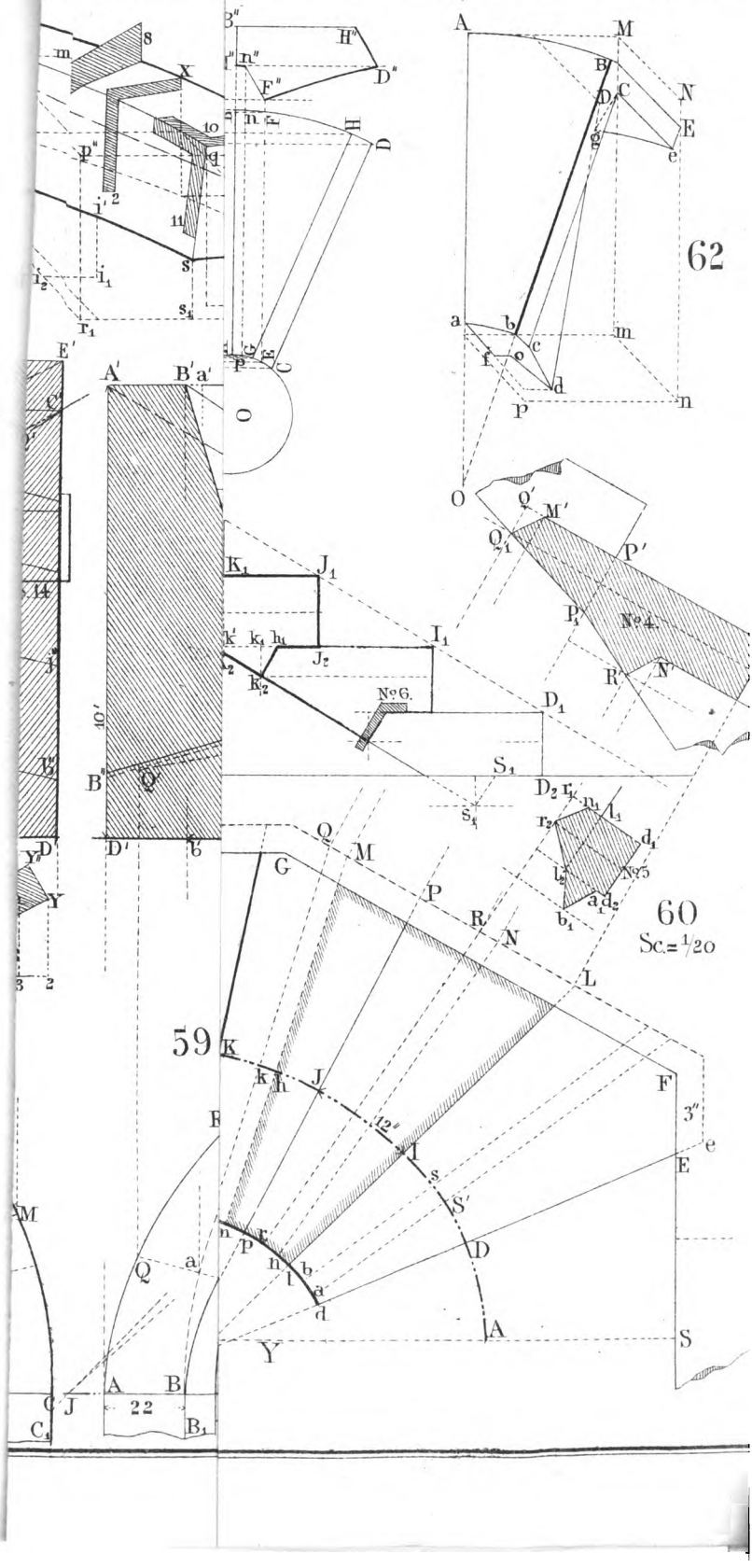
I. *The Projections.* 1°. — *Let a vertical line at* O, Plate X., Fig. 73, be the axis of four concentric vertical cylinders generated by the revolution about this axis of vertical lines at A, *a*, *b*, and B, where OA is 27 ft. 6 in.; OB, 12 ft. 6 in.; AB 15 feet; and *ab* 9 feet, making the thickness of the walls A*a* and B*b*, 3 feet.

Let the circular gallery between the walls be covered by an arch, whose intrados is the half annular torus generated by the revolution of the vertical semicircle *aob*, of 4 ft. 6 in. radius, shown at ac_2b on the plane V. Let this gallery be divided into seven equal compartments, one of which is bounded by the arc AD, of 24.68 ft., and the radials, OA and OD.

This arc, AD, may be accurately determined by laying off any suitable fractional part, as $\frac{1}{n}$ of it, n times. Or, from a table of sides of inscribed regular polygons, we can find the length (here 23.86 ft.) of the chord equal to one side of the regular polygon (here a heptagon), indicated by the number of compartments.

Setting off each way from A, D, etc., 3 ft. on the outer circumference, as at AE and AF, and drawing the radii, as OE and OF, we have the piers, as shown at E*ef*F, and I*ih*H; and the area GFIJ which is to be covered by the radiant arch. This arch is here represented as closed at its ends by twelve inch walls; which, with the piers, form alcoves.

2°. *The intrados of the radiant arch* will, in any case, naturally be a right conoid (114), extending from OF to OG, and whose springing plane will be the same as that of the annular arch; viz., the horizontal plane containing the diameter *ab*. The conoid will then be generated by the straight line OF,


62
60
Sc. = 1/20
59
No 4.
No 5.
No 6.
40'
14
22
12"

moving so as to be parallel to the springing plane as a plane director, while moving upon the vertical line at O, and some curve of a height equal to oc_2, and included symmetrically between OF and OG.

3°. *Two systems.* — At this point, there is a choice between the two methods, one or the other of which would be most naturally chosen.

First. The curved directrix may be the ellipse whose transverse axis is the *chord* of any arc, as oo_1, or GF, included between OG and OF, and whose semi-conjugate axis is a vertical, equal to oc_2 at the middle point of such chord.

Second. The curved directrix, may, instead, be the curve of double curvature, formed by wrapping upon some of the vertical cylinders, as DGA, a semi-ellipse whose transverse axis is the true length of the corresponding *arc*, as GF, and whose semi-conjugate axis equals oc_2. as before.

4°. *Adopting the second system.* — KG_1K_1 is the curved directrix of the conoidal intrados; where KG_1, tangent to GKF at K, equals the *arc* KG, and KK_1 perpendicular to KG_1 equals oc_2. The method of concentric circles on the given *axes*, is adopted, as shown, for constructing this eclipse, on account of its convenience in affording, at the same time, an easy construction of any desired tangents, as also shown.

5°. *The groin curves.* — These are found by auxiliary horizontal planes, taken for convenience though the inner extremities a_1, etc., of the radial joints of the annular arch, where ac_2b is divided here into five equal parts. Each plane will then cut two horizontal circles, centred at O, from the torus, and two elements from the conoid, whose intersections with the circles will be four points of the groin curves. Thus g, f, i, j, are the points determined by the springing plane; c_1 the apex of the groin, is the intersection of the circle Oo with the element KO. Then, for example, making b_1'', at the height, $b_1''x_2, = b_1q$; and $Kx_1, = Kx_2 =$ KM, the two circles with radii, Op and Oq, will intersect the corresponding elements Ox_1 and OM, at the four points, l, l_1, m, and m_1 of the groin curves.

6°. *Nature of the horizontal projections of the groin curves.*— Each of those just found is an arc of a spiral of Archimedes (130), for, from the properties of ellipses having an axis in each equal ($oc_2 = KK_1$) we have,

$$oa_3 : op :: KX_2 : Kx_2;$$

or, by the substitution of equal terms,

$$n_2n : m_2m :: KN : KM.$$

That is, the increments, n_2n and m_2m, of the radius vectors, are proportional to the corresponding increments, KN and KM of the arc which marks their angular movement.

The pole and the initial line. — O, the given intersection of the radius vectors, is the *pole;* and, knowing the character of the curve, find a fourth proportional to jg (= ab) GF (= $2G_1K$) and jO (= bO) and lay it off from G to the *left*, or from F to the *right*, on the circle OK, and points will be found, where the radii to O will be the initial lines of the spirals, fc_1j and gc_1i, respectively; and hence, tangent to them at O.

The portions of these spirals beyond g and f, and within i and j, as at jCO, being projections of no actual lines of the structure, are called *parasites.*

7°. *The tangentoid system.* — Turning to this (3°) for a moment, for comparison, suppose that the vertical ellipse, of transverse axis, gf, and vertical semi-conjugate axis at o_2, = oc_2, had been chosen as the curved directrix of the conoid. Then, as o_1 and o_2 are at equal heights, lines parallel to o_1o_2, as y_1y_2, would give elements; circular, with radius Oy_1, through y_1 on the torus, and straight at y_2O on the conoid, which would meet as at y, a point of the curve of intersection of the torus and the new conoid; a curve whose horizontal projection, gyc_1, etc., would be an arc of a tangentoid spiral (133). For evidently the increments, as o_1y_1, y_1g, etc., of the radius vectors, are proportional to the increments, as o_2y_2, y_2g, etc., of the tangents of the angles made by these radius vectors with Oo_2 as an initial line (134).

8°. *Section and joints of the annular arch.* — Let the figure $Av_1V_1v_3B$ be the section of the annular arch. By revolution about the vertical axis at O, this figure, being in the vertical plane on OA, will generate the volume of masonry, covering the annular arch; and its radial lines, as a_1v_1, will generate the coursing surfaces, or beds, of the voussoirs. These beds will obviously be conical surfaces, as at $N'na_3v$, all having the vertical line at O for their common axis. The heads, or transverse joints, as at YZ, are in vertical planes through O.

9°. *The joints of the conoidal arch.* — The bed surfaces

should, strictly, be normal to the conoidal intrados, along the elements, as MM_1, which are the coursing joints. They will therefore be *hyperbolic-paraboloids* (117).

But at the highest element, the conoidal surface is developable along one element, KO, since a horizontal tangent plane will evidently there be tangent all along that element. Hence, the *normal* surface on KO will also be a plane, and the normal surfaces on elements near KO will therefore be very nearly plane. Hence, the bed surface on NN_1 may properly be plane.

10°. *Construction of the warped bed on* MM_1. — Assume at V_2 a vertical plane of projection, L′T′, perpendicular to MM_1; on which MM_1 is therefore projected in the point m', at the height $L'm' = pa_1 = b_1''x_2$ on V_1. Drawing the tangent $b_1''T''$, as shown, make MT, tangent at M, equal to x_2T'', and by projecting T at T′, $m'T'$ will be its vertical projection. Then (116) TO will be an element of one generation, and the horizontal trace of the *tangent* hyperbolic paraboloid, generated by the motion, parallel to H, of MO, upon MT — $m'T'$ and O — L′M′. Hence, mt, m_1u, M_1U, and Ss, are elements of the other generation of the same *tangent* surface; and $m't'$, $m'u'$, etc., are their vertical projections.

Taking, now, a horizontal plane, $M's_1'$, at a height, L′M′, equal to v_1q_1; perpendiculars $m'R'$, $m'r'$, etc., to $m'T'$, $m't'$, etc., and limited by $M's_1'$, will be those elements of the *normal* hyperbolic paraboloid, which are the lines MT — $m'T'$, etc., revolved 90° about MM_1 — m' as an axis (117). Hence, project R′ on MT at R; r' on mt at r, etc., and MmM_1R_1rR will be the horizontal projection of the required normal bed on MM_1.

The actual limits of this bed are at its intersections mp_2Q, and m_1Q_1, with the corresponding conical beds, Qq_1pm, and Q_1m_1q, of the annular arch. These are found as follows: Assume any intermediate point P on the joint a_1v_1, make $r''s''$ at the height from L′T′, equal to Pp_1, and project r'' at r''' on MT; z' at z, on mt; s'' at s''', etc., and $r'''zs'''$ will be the intersection of the normal joint on MM_1, with the plane $r''s''$; limited at p_2 by p_1p_2, the corresponding horizontal section of the torus. Therefore, $MRQp_2m$ is a definite normal joint. $M_1m_1Q_1R_1$ is another.

11°. *The outer edge,* RR_1, *of the warped bed on* MM_1 *is a hyperbola.*—This we know from the properties of the normal surface, since any plane which cuts the intersection of the two

plane directors cuts the surface in a hyperbola; and as the *tangent* hyperbolic paraboloid on MM_1, revolves 90° on MM_1 to become the *normal* one, its former plane director, of the generation MM_1, which is **H**, becomes a *vertical* plane, parallel to MM_1. The other plane director is perpendicular to MM_1. The intersection of the two is thence a vertical line, which accordingly cuts the horizontal plane $M's'_1$, containing the curve RR_1, which is therefore a hyperbola.

Otherwise; by direct demonstration. The triangles $T'L'm'$ and $m'M'R'$, and other like pairs, are similar, and give

$$L'T' : M'm' :: L'm' : M'R'$$

Also, $$L't' : M'm' :: L'm' : M'r'$$

whence $$L'T' \times M'R' = L't' \times M'r'.$$

That is, $$MT \times MR = mt \times mr.$$

Substituting for MT and mt, the proportionals to them, MO and mO, we have

$$MO \times MR = mO \times mr;$$

which may be written, $$xy = x'y';$$

if the curve be referred to OM, and a perpendicular, OO_2, to it at O, as axes.

At the point equidistant from the axes,

$x' = y'$, and putting $x'y' = k^2$, we have $xy = k^2$; which is the equation of the *hyperbola*, referred to its asymptotes.

At the point s_1, for example, $x = OS$, $y = Ss_1$.

The mean proportional to these is $S1 = k$; whence 2, the vertex, is found, as shown, on the transverse axis, O2, of the curve, which bisects the angle, SOO_1, between the asymptotes.

This being here a right angle, the hyperbola is of the form called equilateral.

12°. *Construction of an approximate plane-normal joint*, on NN_1. — The point N is symmetrical with X_1, hence the tangent at X_1 is symmetrical with that at N. Then draw the tangent b''_2k. Make V_1 tangent to the cylinder DKA at X_1, then X_1x $(= X_2K)$ will be the horizontal projection of this tangent. But b''_2k being straight, and oblique to the elements, as Kk of the cylinder, will, when wrapped upon the cylinder DKA, be transformed into a helix, whose projection, and that of its tangent, on V_1, will in the vicinity of k, be sensibly b''_2k produced. Hence, make *arc* $X_1x'' = X_1x$, project x'' on b''_1k produced, and $x''k_1$, will be the true height at which the tangents at $X_1(b_2)$ and at N, will pierce the plane OK.

This found, take $x'v'$ for the ground line of a new vertical plane $\mathbf{V}_3$; project N upon it at the height a_3n' ($= a_3a_2$), and x at k_1 at the height $x''k_1$; when $n'k_1$ will be the element at Nn' of the *tangent* hyperbolic paraboloid on NN_1; and $n'V'$ perpendicular to it, an element of the *normal* hyperbolic paraboloid on NN_1, where $v'V' = vV$. Hence, projecting V' at N_2 (on the tangent at N) N_2N'', parallel to NN_1, will be the trace, on the horizontal plane $V'V''$ of the top of the annular arch, of the tangent plane at Nn' to the *normal* hyperbolic paraboloid; that is, of the approximate plane joint.

In fact, two such planes should have been similarly made; one tangent at the middle point of Nn, for the bed $NN_2N'n$; and one at the middle point of N_1n_1, for the bed N_1n_1N''.

II. *The Directing Instruments.* — After the previous thorough working up of all the lines and surfaces composing the required structure, these can now be summarily described.

Besides Nos. 1 and 2 (7), there should be, for the stone $N'''N'na_3q_1$, for example, —

No. 3. The pattern $N'''N'vq_1$, of the horizontal plane top.

No. 4. The pattern of the cylindrical back, which, to the the right of Qt, is temporary. In No. 4, $q_36 = q_2N'''$; $b_46 = n_1V'$, both in length and direction; $b_44 = X_3b_2''$; 4 5, and $5b_3'$ are respectively equal to the like spaces on the plan; and $q_3q_4 = V_1v_1$. No. 5 is a pattern of the end, $VV_1v_1a_1a_2$, in the arch.

No. 6 is the development of the conical bed, Qq_1pm. The vertex of this cone is found after revolution, at O', the intersection of OO' perpendicular to OA, with v_1a_1 produced. Hence with O' as a centre draw $a_1m_3 = pm$, and $v_1Q_1 = q_1Q$; likewise $PP_1 = p_1p_2$, and $Q_1v_1a_1m_3$ will be the required pattern.

No. 7 is the pattern found in the same way as was No. 6, of the conical bed $N'va_3n$.

No. 8, which could exist only for a plane bed, is the pattern N_3N_44n, of the plane joint $N'''N'n4$; hence $N_44 = V'n'$.

Nos. 9 and 10, bevels giving respectively the angles V_1Va_2, and Va_2a_1, will be useful in determining the *relative position* of the top, and the conical bed $N'va_3n$; and of this bed and the annular intrados; both 9 and 10, being held in meridian planes of the torus.

No. 11, set to the angle $n'V'n''$, will, in like manner, serve to fix the proper position of $N'N'''4n$, relative to the plane top.

III. *Application.* — Choosing a sufficient block, bring it provisionally to the form of a prism, whose base is na_3q_24, except that the vertical cylindrical face on *na* need not be wrought.

Then, mark the intended top by No. 3, and the back by No. 4. The lines thus given, with the form of the end given by No. 5, will guide the cutting out of the portion Qtq_1q_2.

The lower conical bed can then be directed from the end, and from the vertical surface on V_1v_1, and shaped by No. 6. The upper conical bed will then be determined by No. 9 and No. 7.

The remaining work is obvious enough, the construction of No. 4 showing the ordinates from the base of the provisional prism in the plane $4b_3$ to the edges $n4$ and na_3.

137. *St. Giles' Screw.* — This term is applied to circular stairs with solid central core, or column, where the surface seen overhead in ascending the stairs is a double-curved helicoidal surface, such as would be generated by the section, $AaV'Bb$, of an annular arch, Pl. X., Fig. 73, were it to move so that every point of it should describe a helix about the vertical axis at O; the course of stones generated by Bbb_1v_3 being solid with the core. The coursing surfaces of such an arch would be oblique helicoids, generated by a_1v_1, etc.; and the method of cutting of the stone may be sufficiently understood from the cutting of the voussoirs of the oblique arch.

EXAMPLES. — 1°. Make the full construction of Fig. 73 on the *tangentoid* system.

2°. Make drawings of a key-stone, extending both ways from c_1 in each arch.

3°. Also of an inner pier groin stone, as $M_1m_1Q_1$.

4°. Construct the plane joints indicated in 12°.

5°. Make the necessary illustrative drawings of the double-curved arched covering of circular stairs, known as the St. Giles' screw (137). The coursing joints are oblique helicoids, generated by radii of the semicircle ac_2b, Fig. 73. The heading joints are in planes perpendicular to the helix generated by the highest point, c_2, of the same semicircle. The coursing joints thus generated are not quite normal to the double-curved arch surface, but are nearly enough so. To be perfectly so, they should be generated by normals to that intrados, that is, by lines, not only perpendicular to the tangents to the semicircle, as at its points of division, etc., but also to the tangents to the helices at the same points. The construction would be too laborious, and thence more apt to be inexact.

THE END.